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Valuation of Commodity Derivatives in a New Multi-Factor Model

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Abstract. This paper extends existing commodity valuation models to allow for stochastic volatility and simultaneous jumps in the spot price and spot volatility. Closed-form valuation formulas for forwards, futures, futures options, geometric Asian options and commodity-linked bonds are obtained using the Heston (1993) and Bakshi and Madan (2000) methodology. Stochastic volatility and jumps do *not* affect the futures price at a given point in time. However, numerical examples indicate that they play important roles in pricing options on futures.

Keywords: Asian options, commodity derivatives, random jumps, stochastic volatility.

JEL classification: G13

The past decade or so has seen a proliferation of financial instruments linked to the price of commodities, such as futures, futures options and commodity-linked bonds. With the crude oil price tripled between late 1998 and early 2000, there is now a resurgent interest in commodity risk management. Valuation of commodity derivatives, which are major vehicles for commodity hedging, is becoming an increasingly important problem in financial economics.

The first generation of commodity contingent claims models[1] assume that all the uncertainty is summarized in one factor: the spot price of the commodity. It soon became apparent, however, that more factors are needed to properly value commodity contingent claims. In their two-factor model, Gibson and Schwartz (1990) assume that the spot price and the spot convenience yield follow a joint stochastic process. In his presidential address, Schwartz (1997) develops a three-factor model in which interest rates, in addition to convenience yields and spot prices, are also stochastic. Hillard and Reis (1998) introduce jumps in the spot price of the commodity and use the initial term structure of interest rates to eliminate the market price of interest rate risk in their pricing equation. Miltersen and Schwartz (1998) propose a general framework of pricing commodity futures options using the Heath, Jarrow and Morton (1992) methodology. However, their model is not useful in pricing futures because they take the entire term structure of futures prices as given. It is striking that all of the above models assume that the volatility of spot prices is constant. By now it is a common place observation that the volatility of financial returns (stocks, bonds and currencies) changes over time. What is special about commodities?

Yan (2001) examines the commodity return distributions using gold and crude oil data. Several important characteristics of commodity returns were uncovered in his study. First, commodity returns are leptokurtic. Second, the volatility of commodity returns changes randomly over time. Third, there exist simultaneous abrupt changes in price and volatility. Fourth, commodity options display "volatility smiles." Finally, unlike that of stock indices, commodity return distributions are not negatively skewed and are not characterized by asymmetric volatility (volatility goes up more when price goes down).

To capture the above-mentioned features of commodity returns, I propose a model that incorporates stochastic convenience yields, stochastic interest rates, stochastic volatility and simultaneous jumps in the spot price and volatility. Closed-form solutions are obtained for a variety of commodity derivatives, including futures, forwards, futures options, geometric Asian options and commodity-linked bonds. The futures pricing formula shares a similar feature with existing models in that it is exponentially linear in the factors. The futures price is *not* a function of either spot volatility or jumps or their associated parameters. This is because the futures price is the expected future spot price (the *first* moment) under the risk neutral probability. While stochastic volatility and jumps affect the higher-order moments of the terminal spot price distribution, they do not alter the first moment. Closed-form futures option prices are obtained using the technique developed by Heston (1993) and Bakshi and Madan (2000). Numerical examples show that stochastic volatility and jumps both play important roles in pricing futures options.

The remainder of the paper is organized as follows. The proposed valuation model is presented in Section 1. Closed-form formulas for futures and futures options are derived. Section 1 also briefly discusses the estimation and implementation of the model. Section 2 shows how to price geometric Asian options and a simple class of commodity-linked bonds within the proposed framework. Section 3 concludes.

1. The Valuation Model

1.1. The Setup

The standard modeling procedure for the purpose of derivatives pricing typically involves the following steps. First of all, researchers specify the stochastic process followed by state variables in the objective measure. Secondly, suitable assumptions about the market prices of risks are made. The objective measure is then adjusted by the market price of risks to obtain the risk neutral measure. Finally, derivative prices are obtained by computing the expected discounted future payoff under the risk neutral measure. To save space, I will specify from the outset a stochastic structure under the risk neutral probability measure. As shown by Harrison and Kreps (1979) and Harrison and Pliska (1981), under very general conditions the absence of arbitrage opportunities implies the existence of a risk neutral probability measure. Under this measure the instantaneous expected rate of return of any financial asset that has a positive price is equal to the instantaneous riskless rate. The following assumptions are maintained.

Assumption 1 *Trading takes place continuously.*

Assumption 2 *There are no transactions costs, taxes and short sale constraints.*

Assumption 3 *The dynamics of commodity prices are given by the following stochastic differential equation*

$$dS/S = (r - \delta - \lambda\mu_J)\,dt + \sigma_S\,d\omega_1 + \sqrt{V}\,d\omega_2 + J\,dq \tag{1}$$

$$\text{Prob}\,(dq = 1) = \lambda\,dt \tag{2}$$

$$\ln(1 + J) \sim \mathcal{N}\big(\ln(1 + \mu_J) - \tfrac{1}{2}\sigma_J^2,\ \sigma_J^2\big). \tag{3}$$

Assumption 4 *Spot interest rates follow the square-root process*

$$dr = (\theta_r - \kappa_r r)\,dt + \sigma_r\sqrt{r}\,d\omega_r. \tag{4}$$

Assumption 5 *Spot convenience yields follow the Ornstein-Uhlenbeck (OU) process*

$$d\delta = (\theta_\delta - \kappa_\delta\delta)\,dt + \sigma_\delta\,d\omega_\delta. \tag{5}$$

The convenience yield includes both the reduction in cost of acquiring inventory and the value of being able to profit from temporary local shortage of the commodity. Brennan (1991) proposes that the convenience yield follows an exogenously specified Markov process. This autonomous convenience yield may be regarded as the *reduced form* of a more general model in which the convenience yield is endogenously determined by production, consumption and storage decisions.

Assumption 6 *The spot volatility follows a square-root jump-diffusion process*

$$dV = (\theta_V - \kappa_V V)\,dt + \sigma_V\sqrt{V}d\omega_V + J_V\,dq \tag{6}$$

$$J_V \sim \textit{exponential}(\theta) \qquad \theta > 0. \tag{7}$$

Spot prices and volatilities jump simultaneously. The jump size of volatility follows an exponential distribution. That is, volatility can only jump *up*. This assumption, combined with the assumption of a square-root process, guarantees that spot volatility is always positive.

Assumption 7 *Spot convenience yields and spot volatilities are both correlated with the return process. Correlations between other Brownian Motions are zero*

$$cov(d\omega_1,\, d\omega_\delta) = \rho_1\, dt \tag{8}$$

$$cov(d\omega_2,\, d\omega_V) = \rho_2\, dt. \tag{9}$$

Bakshi, Cao and Chen (1997) and Bates (1996, 2000) have shown that the correlation between the return process and the volatility process is critical in generating skewness and excess kurtosis in the return.[2] This correlation is also able to produce "asymmetric volatility." Specifically, if this correlation is positive, then positive returns are associated with higher volatilities. If this correlation is negative, negative returns are associated with higher volatilities. I expect this parameter to be close to zero for commodities since Yan (2001) finds that "asymmetric volatility" is not a characteristic of commodity returns. Brennan (1991) examines the empirical relationship between inventories of the commodity, spot prices and convenience yields and finds that there is a positive correlation between prices and convenience yields.

The assumption of zero correlations between all other Brownian Motions is necessary for obtaining analytical solutions. For instance, as the instantaneous interest rate and the instantaneous volatility both follow the square root process, a non-zero correlation between $d\omega_r$ and $d\omega_V$ would make the model intractable.

The above commodity return distributional assumptions offer a sufficiently flexible structure that can accommodate most of the desired features. For instance, skewness in the commodity return distributions is controlled by either the correlation ρ_2 or the mean jump size μ_J, whereas the amount of kurtosis is managed by either the volatility of volatility σ_V or the variability of the jump component in the commodity prices. Jumps in volatility can also help generate excess kurtosis.

1.2. Fundamental PDE

The proposed model falls into the class of affine models in that the drift terms and the squared diffusion terms in S, r, δ, and V processes are all linear in state variables. Additionally, the intensity parameter of jumps is constant (hence linear in factors). These assumptions make it possible to obtain closed-form solutions for a variety of commodity derivatives.

Under the risk neutral measure, the instantaneous expected rate of return on any contingent claim with a positive price $F(t,\, \tau)$ is the risk-free rate: $E^Q(dF(t,\, \tau)) = r\, dt$. Define $L(t) = \ln(S(t))$. Expanding $E^Q(dF(t,\, \tau))$ by the generalized Ito's lemma, I obtain the following PDE that $F(t,\, \tau)$ has to satisfy

$$\begin{aligned}
&\tfrac{1}{2}(\sigma_S^2 + V)F_{LL} + \tfrac{1}{2}\sigma_r^2 rF_{rr} + \tfrac{1}{2}\sigma_\delta^2 F_{\delta\delta} + \tfrac{1}{2}\sigma_V^2 VF_{VV} + \sigma_S\sigma_\delta\rho_1 F_{L\delta} + \sigma_V V\rho_2 F_{LV} \\
&\quad + \left(r - \delta - \lambda\mu_J - \tfrac{1}{2}\sigma_S^2 - \tfrac{1}{2}V\right)F_L + (\theta_r - \kappa_r r)F_r + (\theta_\delta - \kappa_\delta\delta)F_\delta + (\theta_V - \kappa_V V)F_V \\
&\quad - F_\tau - rF + \lambda E\{F(t,\, \tau;\, L + \ln(1+J),\, V + J_V) - F(t,\, \tau;\, L,\, V)\} = 0
\end{aligned} \tag{10}$$

subject to security-specific boundary conditions.

1.3. Bond Price

The price of a riskfree discount bond is given in Cox, Ingersoll and Ross (1985):

$$B(t,\ \tau) = \exp[-\zeta_0(\tau) - \zeta_1(\tau)r] \tag{11}$$

where

$$\begin{aligned}
\zeta_0(\tau) &= \frac{\theta_r}{\sigma_r^2}\left\{(\varpi - \kappa_r)\tau + 2\ln\left[1 - \frac{(1-e^{-\varpi\tau})(\varpi-\kappa_r)}{2\varpi}\right]\right\}\\
\zeta_1(\tau) &= \frac{2(1-e^{-\varpi\tau})}{2\varpi - [\varpi-\kappa_r](1-e^{-\varpi\tau})}\\
\varpi &\equiv \sqrt{\kappa_r^2 + 2\sigma_r^2}\,.
\end{aligned}$$

1.4. Futures Price

1.4.1. Futures Pricing Formula

Let $H(t,\ \tau)$ denote the futures price at time t with a time to maturity τ. Cox, Ingersoll and Ross (1981) and others have shown that the futures price is a martingale under the risk neutral measure. Since futures contracts cost nothing to enter, its expected return must be zero: $E^Q[dH(t,\ \tau)] = 0$. Expanding the left hand side using the generalized Ito's lemma, I obtain the following PDE $H(t,\ \tau)$ must satisfy

$$\begin{aligned}
&\tfrac{1}{2}(\sigma_S^2 + V)H_{LL} + \tfrac{1}{2}\sigma_r^2 r H_{rr} + \tfrac{1}{2}\sigma_\delta^2 H_{\delta\delta} + \tfrac{1}{2}\sigma_V^2 V H_{VV} + \sigma_S\sigma_\delta\rho_1 H_{L\delta} + \sigma_V V\rho_2 H_{LV}\\
&\quad + \left(r - \delta - \lambda\mu_J - \tfrac{1}{2}\sigma_S^2 - \tfrac{1}{2}V\right)H_L + (\theta_r - \kappa_r r)H_r + (\theta_\delta - \kappa_\delta\delta)H_\delta + (\theta_V - \kappa_V V)H_V\\
&\quad - H_\tau + \lambda E\{H(t,\ \tau;\ L + \ln(1+J),\ V + J_V) - H(t,\ \tau;\ L,\ V)\} = 0
\end{aligned} \tag{12}$$

subject to $H(t+\tau,\ 0) = S(t+\tau)$. In Appendix 1, I show that $H(t,\ \tau)$ is:

$$H(t,\ \tau) = \exp\{\ln(S) + \beta_0(\tau) + \beta_r(\tau)r + \beta_\delta(\tau)\delta\} \tag{13}$$

where

$$\begin{aligned}
\beta_r(\tau) &= \frac{2(1-e^{-\xi_r\tau})}{2\xi_r - [\xi_r - \kappa_r](1-e^{-\xi_r\tau})}\\
\beta_\delta(\tau) &= \frac{-(1-e^{-\kappa_\delta\tau})}{\kappa_\delta}
\end{aligned}$$

$$\xi_r = \sqrt{\kappa_r^2 + 2\sigma_r^2}$$

$$\beta_0(\tau) = -\frac{\theta_r}{\sigma_r^2}\left[2\ln\left(1 - \frac{(\xi_r - \kappa_r)(1 - e^{-\xi_r \tau})}{2\xi_r}\right) + (\xi_r - \kappa_r)\tau\right] + \frac{\sigma_\delta^2 \tau}{2\kappa_\delta^2} - \frac{\sigma_S \sigma_\delta \rho_1 + \theta_\delta}{\kappa_\delta}\tau$$
$$- \frac{(\sigma_S \sigma_\delta \rho_1 + \theta_\delta)e^{-\kappa_\delta \tau}}{\kappa_\delta^2} + \frac{4\sigma_\delta^2 e^{-\kappa_\delta \tau} - \sigma_\delta^2 e^{-2\kappa_\delta \tau}}{4\kappa_\delta^3} + \frac{\sigma_S \sigma_\delta \rho_1 + \theta_\delta}{\kappa_\delta^2} - \frac{3\sigma_\delta^2}{4\kappa_\delta^3}.$$

The futures price is *not* a function of spot volatility, jumps or their associated parameters. It should be emphasized, however, that this result does *not* imply that stochastic volatility and jumps are unimportant. They are important for capturing the dynamics of spot prices. But they are not important for pricing futures at a given point in time. This is because the futures price is the expectation of the future spot price under the risk neutral measure. While stochastic volatility and jumps affect the higher-order moments of the terminal spot price distribution, they do not alter the first moment. In contrast, it will be shown that stochastic volatility and jumps play critical roles in pricing options. This is because option prices are sensitive to higher-order moments of the distribution of terminal prices.

1.4.2. The Basis

The basis is the difference between the futures price and the cash price. Alternatively, it can be defined as the difference between the log futures price and the log cash price. Using the second definition and the futures pricing formula (13), I find that the basis is:

$$\ln(H(t, \tau)) - \ln(S) = \beta_0(\tau) + \beta_r(\tau)r + \beta_\delta(\tau)\delta. \tag{14}$$

Since both r and δ are stochastic and mean reverting, the basis is also stochastic and mean reverting. In addition, the basis is a function of time to maturity of the futures contract.

1.4.3. Futures Return Volatility

From the futures pricing formula (13), I obtain the instantaneous volatility of future returns:

$$\sigma_H^2(\tau) = \sigma_S^2 + V + \lambda\left[\mu_J^2 + \left(e^{\sigma_J^2} - 1\right)(1 + \mu_J)^2\right] + \beta_r^2 r \sigma_r^2 + \beta_\delta^2 \sigma_\delta^2 + \rho_1 \beta_\delta \sigma_S \sigma_\delta. \tag{15}$$

Since V and r are stochastic, the volatility of futures returns is also stochastic. Equation (15) can be used to examine the validity of Samuelson hypothesis. Samuelson hypothesis states that the futures volatility is higher closer to delivery. In this model, β_r is positive and increasing in τ. β_δ is negative and decreasing in τ. The β_r^2 term and the β_δ^2 term therefore affect the futures volatility in the same direction. β_δ has the opposite effect, however. Hence, whether the futures return volatility is decreasing in time to maturity depends upon

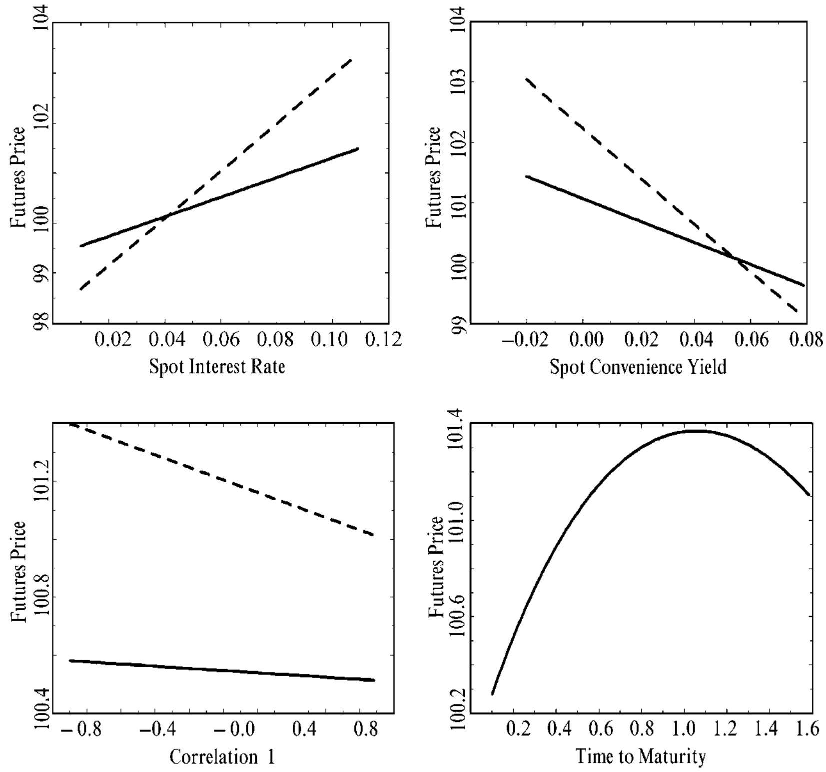

Figure 1. Comparative Statics of Futures Prices. Comparative statics of futures price with respect to the spot interest rate, spot convenience yield, the correlation between the return process and the convenient yield process (ρ_1) and time to maturity. Baseline parameter values are given in Section 2.7. In the first three plots, the dashed line corresponds to the futures with 0.5 years to maturity and the solid line corresponds to the futures with 0.2 years to maturity. Interest rates and convenience yields are annualized. The time to maturity is in years.

the relative magnitude of these two effects. The lower right plot of Figure 1 shows that the futures volatility is first decreasing and then increasing with time to maturity under the parameter values given in Section 2.7.

1.5. Forward Price

By standard arguments [see Duffie (1996)], the forward price is:

$$F(t, \tau) = \frac{E^Q\left[e^{-\int_t^{t+\tau} r(u)du} \times S(t+\tau)\right]}{E^Q\left[e^{-\int_t^{t+\tau} r(u)du}\right]}. \tag{16}$$

It is obvious that if r is uncorrelated with S then the forward price is the same as the futures price. The denominator is the price of the discount bond given in (11). The numerator, which can be computed similarly as the futures price, is the price of a contingent claim that pays $S(t+\tau)$ at $t+\tau$

$$F(t,\,\tau) = \frac{\exp\{L + \varphi_0(\tau) + \varphi_\delta(\tau)\delta\}}{B(t,\,\tau)} \tag{17}$$

where

$$\begin{aligned}
\varphi_\delta(\tau) &= \frac{e^{-\kappa_\delta \tau} - 1}{\kappa_\delta} \\
\varphi_0(\tau) &= \left(\frac{\sigma_S\sigma_\delta\rho_1 + \theta_\delta}{-\kappa_\delta} + \frac{\sigma_\delta^2}{2\kappa_\delta^2}\right)\tau + \frac{-(\sigma_S\sigma_\delta\rho_1 + \theta_\delta)}{\kappa_\delta^2}\,e^{-\kappa_\delta\tau} \\
&\quad - \frac{\sigma_\delta^2}{4\kappa_\delta^3}\,e^{-2\kappa_\delta\tau} + \frac{\sigma_\delta^2}{\kappa_\delta^3}\,e^{-\kappa_\delta\tau} + \frac{3\sigma_\delta^2 - 4\kappa_\delta(\sigma_S\sigma_\delta\rho_1 + \theta_\delta)}{-4\kappa_\delta^3}.
\end{aligned}$$

1.6. Futures Option

Let $C(t,\,\tau)$ denote the price of an *European* call option on the futures contract that matures in $\tilde{\tau} > \tau$, where τ is the time to maturity of the option contract and K is the strike price.[3] The European option is priced as the expected discounted payoffs under the risk neutral measure,

$$C(t,\,\tau) = E^Q\left[e^{-\int_t^{t+\tau} r(u)du} \times \max(H(t+\tau,\,\tilde{\tau}-\tau) - K,\,0)\right].$$

Following Heston (1993), Bates (1996), and Bakshi and Madan (2000), I show in Appendix 2 that $C(t,\,\tau)$ can be decomposed to:

$$C(t,\,\tau) = G(t,\,\tau)\Pi_1(t,\,\tau) - KB(t,\,\tau)\Pi_2(t,\,\tau) \tag{18}$$

where

$$B(t,\,\tau) = E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right)\right\} \tag{19}$$

$$G(t,\,\tau) = E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times H(t,\,\tilde{\tau})\right\} \tag{20}$$

$$\Pi_1(t,\,\tau) = \frac{1}{G(t,\tau)} \times E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times H(t,\,\tilde{\tau}) | H(t,\,\tilde{\tau}) \geq K\right\} \tag{21}$$

$$\Pi_2(t,\,\tau) = \frac{1}{B(t,\tau)} \times E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right)\middle| H(t,\,\tilde{\tau}) \geq K\right\}. \tag{22}$$

$B(t,\ \tau)$ is the discount bond price. $G(t,\ \tau)$ is the price of a contingent claim that pays $H(t+\tau,\ \tilde{\tau}-\tau)$ at $t+\tau$. Π_1 and Π_2 are two risk neutralized probabilities that the option expires in the money. Define the discounted characteristic function of the logarithm of the futures price

$$f(t,\ \tau;\ \phi) \equiv E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times e^{i\phi \ln H(t,\tilde{\tau})}\right\}. \tag{23}$$

Bakshi and Madan (2000) show that $B(t,\ \tau)$, $G(t,\ \tau)$, $f_1(t,\ \tau;\ \phi)$ and $f_2(t,\ \tau;\ \phi)$ (the characteristic functions of Π_1 and Π_2 respectively) are related to $f(t,\ \tau;\ \phi)$ in the following way:

$$B(t,\ \tau) = f(t,\ \tau;\ 0) \tag{24}$$

$$G(t,\ \tau) = f\left(t,\ \tau;\ \frac{1}{i}\right) \tag{25}$$

$$f_1(t,\ \tau;\ \phi) = \frac{1}{G(t,\tau)} \times f\left(t,\ \tau;\ \frac{1}{i}+\phi\right) \tag{26}$$

$$f_2(t,\ \tau;\ \phi) = \frac{1}{B(t,\tau)} \times f(t,\ \tau;\ \phi). \tag{27}$$

Hence, the key is to find a closed-form solution for $f(t,\ \tau;\ \phi)$. Since $f(t,\ \tau;\ \phi)$ is the price of a contingent claim that pays $e^{i\phi \ln H(t+\tau,\tilde{\tau}-\tau)}$ at $t+\tau$, it satisfies the fundamental PDE (10) subject to $f(t+\tau,\ 0) = e^{i\phi \ln H(t+\tau,\tilde{\tau}-\tau)}$.

It is shown in Appendix 2 that $f(t,\ \tau;\ \phi)$ is:

$$\begin{aligned} f(t,\tau;\ \phi) = \exp\{&i\phi L + \vartheta_0(\tau) + \vartheta_r r + \vartheta_\delta \delta + \vartheta_V(\tau)V \\ &+ i\phi[\beta_0(\tilde{\tau}-\tau) + \beta_r(\tilde{\tau}-\tau)r + \beta_\delta(\tilde{\tau}-\tau)\delta]\} \end{aligned} \tag{28}$$

where

$$\vartheta_r = \frac{2\left[i\phi - 1 - \frac{1}{2}\sigma_r^2\phi^2\beta_r^2 - \kappa_r i\phi\beta_r\right]\left(1-e^{-\xi_r^*\tau}\right)}{2\xi_r^* - [\xi_r^* - \kappa_r + \sigma_r^2 i\phi\beta_r](1-e^{-\xi_r^*\tau})}$$

$$\vartheta_\delta = \frac{(i\phi + i\phi\beta_\delta\kappa_\delta)(e^{-\kappa_\delta\tau}-1)}{\kappa_\delta}$$

$$\vartheta_V = \frac{i\phi(i\phi-1)\left(1-e^{-\xi_V^*\tau}\right)}{2\xi_V^* - [\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi]\left(1-e^{-\xi_V^*\tau}\right)}$$

$$\xi_r^* = \sqrt{(\sigma_r^2 i\phi\beta_r - \kappa_r)^2 - 2\sigma_r^2\left[i\phi - 1 - \frac{1}{2}\sigma_r^2\phi^2\beta_r^2 - \kappa_r i\phi\beta_r\right]}$$

$$\xi_V^* = \sqrt{(\kappa_V - \sigma_V\rho_2 i\phi)^2 - i\phi(i\phi-1)\sigma_V^2}$$

and $\vartheta_0(\tau)$ is given in Appendix 2.

After obtaining a closed-form solution for $f(t, \tau;\ \phi)$, one can compute $G(t, \tau)$, $f_1(t, \tau;\ \phi)$ and $f_2(t, \tau;\ \phi)$ using (25), (26) and (27). Π_1 and Π_2 are then recovered by Fourier inversion.[4] For $j = 1,\ 2$,

$$\Pi_j(t, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} \times f_j(t, \tau;\ \phi)}{i\phi}\right] d\phi. \tag{29}$$

Finally, the option price is given by (18). The closed-form option pricing formula makes it possible to derive comparative statics and hedge ratios analytically

$$\Delta_H = \frac{\partial C}{\partial H} = \frac{G(t,\tau)}{H(t,\tau)}\,\Pi_1. \tag{30}$$

The above formula is obtained by using the property that the call option price is homogeneous of degree one in H and K

$$\Delta_V = \frac{\partial C}{\partial V} = G(t,\ \tau)\frac{\partial \Pi_1}{\partial V} + KB(t,\ \tau)\frac{\partial \Pi_2}{\partial V} \tag{31}$$

$$\Delta_r = \frac{\partial C}{\partial r} = G(t,\ \tau)\frac{\partial \Pi_1}{\partial r} + \frac{\partial G(t,\tau)}{\partial r}\Pi_1 + KB(t,\ \tau)\left[\frac{\partial \Pi_2}{\partial r} + \zeta_1(\tau)\Pi_2\right] \tag{32}$$

$$\Delta_\delta = \frac{\partial C}{\partial \delta} = G(t,\ \tau)\frac{\partial \Pi_1}{\partial \delta} + \frac{\partial G(t,\tau)}{\partial \delta}\Pi_1 + KB(t,\ \tau)\frac{\partial \Pi_2}{\partial \delta}. \tag{33}$$

$\frac{\partial \Pi_j}{\partial r}$, $\frac{\partial \Pi_j}{\partial V}$, and $\frac{\partial \Pi_j}{\partial \delta}$ can be computed similarly as Π_1 and Π_2 by exchanging integrals and derivatives. $\frac{\partial G(t,\tau)}{\partial r}$, and $\frac{\partial G(t,\tau)}{\partial \delta}$ are easy to compute because $G(t,\ \tau)$ is exponentially linear in r and δ.

1.7. Numerical Examples

In this section, I use numerical examples to examine if stochastic volatility, stochastic convenience yields, stochastic interest rates and jumps have significant impacts on futures and futures option prices. I compare five models.

Model 1: Stochastic prices, stochastic convenience yields, stochastic interest rates, stochastic volatility and jumps.

Model 2: Stochastic prices, stochastic convenience yields, stochastic interest rates and stochastic volatility.

Model 3: Stochastic prices, stochastic convenience yields and stochastic interest rates.

Model 4: Stochastic prices and stochastic convenience yields.

Model 5: Stochastic prices.

The following parameter values used in Model 1 are partly based on Schwartz (1997). $S = 100$, $r = 0.06$, $\delta = 0.03$, $\lambda = 1$, $\mu_J = 0$, $\sigma_S = 0.1$, $V = 0.04$, $\theta_r = 0.015$, $\kappa_r = 0.25$, $\sigma_r = 0.1$, $\theta_\delta = 0.03$, $\kappa_\delta = 1$, $\sigma_\delta = 0.2$, $\theta_V = 0.08$, $\kappa_V = 2$, $\sigma_V = 0.1$, $\sigma_J = 0.05$, $\theta = 0.01$, $\rho_1 = 0.8$, $\rho_2 = 0$. τ can take on two values 0.2 years or 0.5 years. $\tilde{\tau} - \tau$ is either 0.05 years or 0.25 years. Whenever appropriate, the parameter values are annualized. For instance, a spot interest rate of 0.06 should be interpreted as 6% per year. A jump intensity of 1 indicates that there is on average one jump per year. For ease of interpretation, I fix the spot price to 100. The futures price, strike price and option price can be interpreted as percentages of the spot price. The possible values for K are 70, 80, 90, 100, 110, 120, 130. The parameter values of Models 2, 3, 4, 5 are derived from Model 1 parameters while keeping the *total volatility* unchanged.[5]

Figure 1 graphs how the futures price under Model 1 changes with the spot interest rate (r), the spot convenience yield (δ), the correlation between spot returns and convenience yields (ρ_1) and the time to maturity (τ). Not surprisingly, the futures price is increasing in r and decreasing in δ. The futures price is decreasing in ρ_1. This is because higher ρ_1 generates stronger mean reversion effect in commodity prices. The futures price is hump-shaped over τ under the above parameter values. In general, this model can generate monotonically increasing, decreasing, hump- and bell-shaped *term structure* of futures prices.

Tables 1 and 2 compare futures *call* option prices for Models 1–5 under different maturities and strike prices. Tables 1 and 2 also report the ratios between option prices under Models 2–5 and the option prices under Model 1. Assuming Model 1 is the right model, these ratios essentially give the percentage pricing errors.[6] As can be seen in Tables 1 and 2, jumps, stochastic volatility and stochastic convenience yields are all important factors in determining option prices. The stochastic interest rate is the least important factor, especially for short term options. Not surprisingly, the percentage pricing error is much higher for the Out-of-The-Money (OTM) and At-The-Money (ATM) options. Generally speaking, Model 5 tends to overprice options while other models tend to underprice, relative to Model 1.

Figure 2 gives the comparative statics of Model 1 futures call option prices. The option prices are most sensitive to the spot volatility. The futures option price is increasing in spot interest rates. Additionally, the futures option price is decreasing in the spot convenience yield. The volatility of volatility is not a significant factor.

1.8. Implementation and Estimation of the Model

As the convenience yields and volatility are not traded assets, one needs to estimate the market prices of risks in order to implement the proposed model. In principle, one can estimate this model using futures data and/or futures option data. Like most continuous-time models in the literature, the exact conditional density function is unknown for this model. Hence, direct maximum likelihood estimation is not feasible. I propose a two-step estimation procedure, which takes advantage of the unique features of the model. In the first step, futures and interest rate data are used to estimate the parameters associated with the spot price, spot interest rates and spot convenience yield processes, i.e., equations

Table 1. Comparison of European Futures Call Option Prices I

Models 1–5 are described in Sections 1.1 and 1.7. τ is the time to maturity of the futures options and $\tilde{\tau}$ is the time to maturity of the futures, both in years. Parameter values are given in Section 1.7. The numbers in parentheses are the ratios of the option prices with respect to the corresponding option prices of Model 1.

Strike	Model 1	Model 2	Model 3	Model 4	Model 5
			$\tau = 0.2, \tilde{\tau} = 0.25$		
70	30.35	30.34 (0.99)	30.33 (0.99)	30.33 (0.99)	30.37 (1.00)
80	20.50	20.49 (0.99)	20.48 (0.99)	20.48 (0.99)	20.53 (1.00)
90	11.20	11.16 (0.99)	11.15 (0.99)	11.15 (0.99)	11.57 (1.03)
100	4.29	4.25 (0.98)	4.24 (0.98)	4.24 (0.98)	4.42 (1.02)
110	1.07	1.03 (0.96)	1.02 (0.95)	1.02 (0.95)	1.15 (1.06)
120	0.18	0.16 (0.89)	0.15 (0.82)	0.15 (0.82)	0.19 (1.04)
130	0.02	0.02 (0.76)	0.01 (0.19)	0.01 (0.21)	0.01 (0.55)
			$\tau = 0.2, \tilde{\tau} = 0.45$		
70	30.80	30.78 (0.99)	30.77 (0.99)	30.77 (0.99)	30.88 (1.00)
80	20.94	20.92 0.99)	20.91 (0.99)	20.91 (0.99)	21.04 (1.00)
90	11.55	11.50 (0.99)	11.49 (0.99)	11.49 (0.99)	11.72 (1.01)
100	4.45	4.38 (0.98)	4.37 (0.98)	4.37 (0.98)	4.71 (1.05)
110	1.10	1.04 (0.94)	1.03 (0.93)	1.03 (0.93)	1.56 (1.41)
120	0.18	0.15 (0.86)	0.14 (0.78)	0.14 (0.78)	0.22 (1.20)
130	0.02	0.02 (0.67)	0.01 (0.10)	0.01 (0.12)	0.02 (0.72)
			$\tau = 0.5, \tilde{\tau} = 0.55$		
70	30.64	30.60 (0.99)	30.56 (0.99)	30.55 (0.99)	30.72 (1.00)
80	21.59	21.51 (0.99)	21.16 (0.98)	21.16 (0.98)	21.40 (0.99)
90	13.03	12.89 (0.98)	12.85 (0.98)	12.84 (0.98)	13.22 (1.01)
100	6.87	6.69 (0.97)	6.65 (0.96)	6.65 (0.96)	7.10 (1.03)
110	3.11	2.96 (0.94)	2.91 (0.93)	2.90 (0.93)	3.30 (1.05)
120	1.54	1.13 (0.73)	1.07 (0.69)	1.07 (0.69)	1.32 (0.85)
130	0.44	0.38 (0.85)	0.31 (0.70)	0.31 (0.70)	0.45 (1.01)

(1)–(5). This step can be done using Kalman filter, as done in Schwartz (1997). In the second step, one can use the option data to estimate the parameters associated with volatilitity and jump processes as done in Bates (2000) and Bakshi, Cao and Chen (1997). This above separation is efficient because (a) the stochastic volatility and jump parameters do not enter the future pricing formula and (b) option prices are most sensitive to volatility and jump parameters.

2. Other Commodity Derivatives

The proposed commodity valuation model is capable of generating closed-form solutions for a variety of exotic commodity derivatives. In the following, I demonstrate this by showing how to price geometric Asian options and a simple class of commodity-linked bonds within the proposed model.

Table 2. Comparison of European Futures Call Option Prices II

Models 1–5 are described in Sections 1.1 and 1.7. τ is the time to maturity of the futures options and $\tilde{\tau}$ is the time to maturity of the futures, both in years. Parameter values are given in Section 1.7. The numbers in parentheses are the ratios of the option prices with respect to the corresponding option prices of Model 1.

Strike	Model 1	Model 2	Model 3	Model 4	Model 5
			$\tau = 0.5, \tilde{\tau} = 0.75$		
70	31.08	31.03 (0.99)	30.98 (0.99)	31.02 (0.99)	31.52 (1.01)
80	21.69	21.60 (0.99)	21.54 (0.99)	21.54 (0.99)	21.88 (1.01)
90	13.35	13.18 (0.98)	13.13 (0.98)	13.12 (0.98)	13.63 (1.02)
100	7.06	6.84 (0.96)	6.80 (0.96)	6.79 (0.96)	7.39 (1.05)
110	3.21	3.01 (0.93)	2.96 (0.92)	2.95 (0.92)	3.47 (1.08)
120	1.57	1.14 (0.72)	1.08 (0.68)	1.07 (0.68)	1.41 (0.90)
130	0.45	0.38 (0.83)	0.31 (0.68)	0.31 (0.68)	0.49 (1.07)
			$\tau = 1, \tilde{\tau} = 1.05$		
70	31.33	31.18 (0.99)	31.04 (0.99)	31.02 (0.99)	31.45 (1.00)
80	22.84	22.60 (0.98)	22.45 (0.98)	22.44 (0.98)	23.00 (1.01)
90	15.58	15.24 (0.97)	15.09 (0.96)	15.07 (0.96)	15.78 (1.01)
100	9.95	9.54 (0.95)	9.40 (0.94)	9.38 (0.94)	10.15 (1.02)
110	5.98	5.58 (0.93)	5.43 (0.90)	5.42 (0.90)	6.13(1.03)
120	3.42	3.08 (0.90)	2.91 (0.85)	2.90 (0.85)	3.49 (1.02)
130	1.87	1.62 (0.86)	1.43 (0.76)	1.43 (0.76)	1.87 (1.00)
			$\tau = 1, \tilde{\tau} = 1.25$		
70	31.78	31.61 (0.99)	31.46 (0.99)	31.45 (0.98)	31.92 (1.00)
80	23.27	22.99 (0.98)	22.84 (0.98)	22.82 (0.98)	23.44 (1.01)
90	15.95	15.56 (0.97)	15.41 (0.96)	15.39 (0.96)	16.16 (1.01)
100	10.24	9.79 (0.95)	9.64 (0.94)	9.62 (0.93)	10.45 (1.02)
110	6.20	5.75 (0.92)	5.60 (0.90)	5.57 (0.90)	6.35 (1.03)
120	3.56	3.19 (0.89)	3.01 (0.84)	3.00 (0.84)	3.64 (1.02)
130	1.97	1.68 (0.85)	1.49 (0.75)	1.49 (0.75)	1.96 (1.00)

2.1. Asian Options

The Asian option based on the geometric average of futures prices can be evaluated as in Bakshi and Madan (2000). Recall that the futures price is exponentially linear in state variables. This turns out to be critical to generate a closed-form solution for geometric Asian options.

The geometric Asian call can be expressed as:

$$\hat{C}(t,\,\tau) = E_t^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times \max\left(\exp\left\{\frac{1}{t+\tau}\int_0^{t+\tau} \ln H(u,\,\tilde{\tau})du\right\} - K,\,0\right)\right\} \tag{34}$$

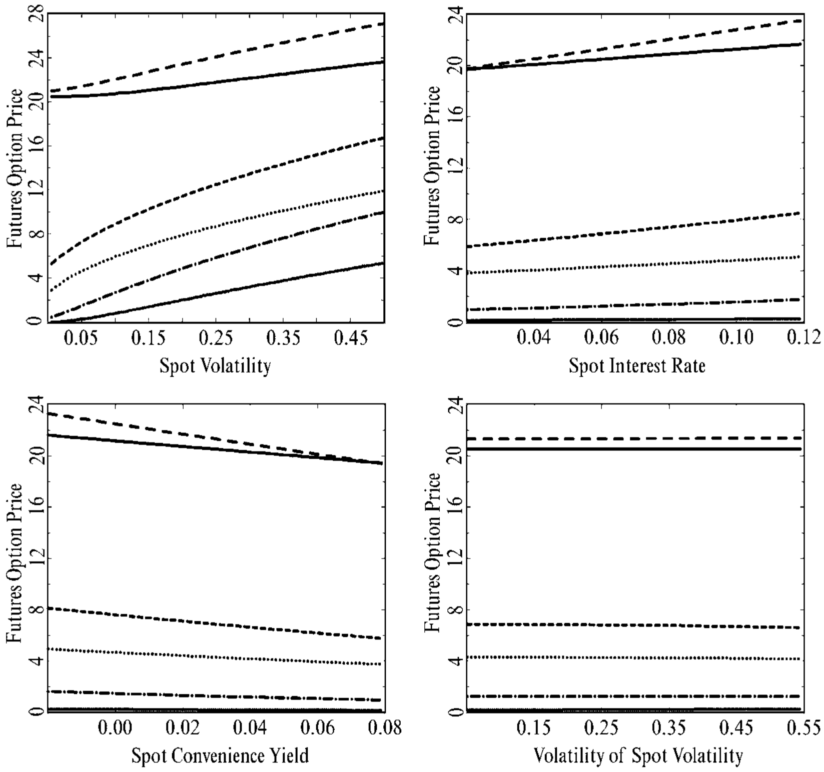

Figure 2. Comparative Statics of Futures Call Option Prices. Comparative statics of futures call option prices with respect to the spot volatility, spot interest rate, spot convenience yield and the volatility of spot volatility. Baseline parameter values are given in Section 1.7 except the strike price and the years to maturity, which are given below. In each of the four plots, the first line from top is associated with a time to maturity of 0.5 years and a strike price of 80. The second line is associated with a time to maturity of 0.2 years and a strike price of 80. The third line is associated with a time to maturity of 0.5 years and a strike price of 100. The fourth line is associated with a time to maturity of 0.2 years and a strike price of 100. The fifth line is associated with a time to maturity of 0.5 years and a strike price of 120. The sixth line is associated with a time to maturity of 0.2 years and a strike price of 120.

with the exercise region being:

$$\int_t^{t+\tau} \ln H(t,\, \tilde{\tau})\, du \;\geq (t+\tau)\ln K - \int_0^t \ln H(u,\, \tilde{\tau}) du.$$

The discounted characteristic function for the remaining log futures price uncertainty is:

$$h(t,\, \tau;\; \phi) \equiv E_t\left\{ \exp\left(-\int_t^{t+\tau} r(u) du \right) \times \exp\left(i\phi \int_t^{t+\tau} \ln H(u,\, \tilde{\tau}) du \right) \right\} \qquad (35)$$

then

$$\hat{C}(t,\ \tau) = N(t,\ \tau) \times \exp\left\{\frac{G(t)}{t+\tau}\right\} \times \Pi_1(t,\ \tau) - KB(t,\ \tau)\Pi_2(t,\ \tau) \tag{36}$$

where

$$G(t) \equiv \int_0^t \ln F(u,\ \tilde{\tau})du$$

and for $j = 1,\ 2$,

$$\Pi_j(t,\ \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\exp(-i\phi(t+\tau)\ln K + i\phi G(t))\frac{h_j(t,\tau;\ \phi)}{i\phi}\right\}d\phi$$

with

$$B(t,\ \tau) = h(t,\ \tau;\ 0) \tag{37}$$

$$N(t,\ \tau) = h\left(t,\ \tau;\ \frac{1}{i(t+\tau)}\right) \tag{38}$$

and

$$h_1(t,\ \tau;\ \phi) = \frac{1}{N(t,\tau)} \times h\left(t,\ \tau;\ \phi + \frac{1}{i(t+\tau)}\right) \tag{39}$$

$$h_2(t,\ \tau;\ \phi) = \frac{1}{B(t,\tau)} \times h(t,\ \tau;\ \phi). \tag{40}$$

I can rewrite $h(t,\ \tau)$ as:

$$h(t,\ \tau;\ \phi) = E_t^Q \exp\left\{-\int_t^{t+\tau}[r(u) - i\phi\ln H(u,\ \tilde{\tau})]du\right\}. \tag{41}$$

By Feynman–Kac theorem, it satisfies the following PDE:

$$\begin{aligned}&\tfrac{1}{2}(\sigma_S^2 + V)h_{LL} + \tfrac{1}{2}\sigma_\delta^2 h_{\delta\delta} + \tfrac{1}{2}\sigma_V^2 V h_{VV} + \sigma_S\sigma_\delta\rho_1 h_{L\delta} + \sigma_V V\rho_2 h_{LV}\\ &\quad + \left(r - \delta - \lambda\mu_J - \tfrac{1}{2}\sigma_S^2 - \tfrac{1}{2}V\right)h_L + (\theta_r - \kappa_r)h_r + (\theta_\delta - \kappa_\delta)h_\delta + (\theta_V - \kappa_V)h_V\\ &\quad - h_\tau - (r - i\phi\ln H(t,\tilde{\tau}))h + \lambda E\{h(t;\ L + \ln(1+J), V + J_V) - h(t;\ L, V)\} = 0\end{aligned} \tag{42}$$

subject to $h(t+\tau,\ 0) = 1$.

Note that $\ln H(t,\ \tilde{\tau})$ is linear in r, δ and V. Hence we have a PDE with all the coefficients linear in state variables. It can be solved using the Heston (1993) and Bakshi and Madan (2000) method

$$h(t,\ \tau) = \exp\{\gamma_0(\tau) + \gamma_r(\tau)r + \gamma_\delta(\tau)\delta + \gamma_V(\tau)V + i\phi\tau L\} \tag{43}$$

where

$$\gamma_r = \frac{2(i\phi\tau + i\phi\beta_r - 1)(1 - e^{-\eta_r\tau})}{2\eta_r - [\eta_r - \kappa_r](1 - e^{-\eta_r\tau})}$$

$$\gamma_V = \frac{(i\phi\tau + \phi^2\tau^2)(1 - e^{-\eta_V\tau})}{2\eta_V - [\eta_V - \kappa_V + \sigma_V\rho_2 i\phi](1 - e^{-\eta_r\tau})}$$

$$\gamma_\delta = \frac{i\phi\beta_\delta - i\phi\tau}{\kappa_\delta}(1 - e^{-\kappa_\delta\tau})$$

$$\eta_r = \sqrt{\kappa_r^2 - 2\sigma_r^2(i\phi\tau + i\phi\beta_r - 1)}$$

$$\eta_V = \sqrt{(\kappa_V - \sigma_V\rho_2 i\phi)^2 + \sigma_V^2(i\phi\tau + \phi^2\tau)}$$

and γ_0 is given in Appendix 3.

2.2. *Commodity-Linked Bonds*

Consider a simple class of commodity-linked bonds. At maturity $t + \tau$, the issuing company promises to pay a face value F or N units of the commodity whose current price is $S(t)$. *If* we can ignore the default risk, the price of this bond $\hat{B}(t, \tau)$ can be nicely decomposed into two components

$$\hat{B}(t, \tau) = FB(t, \tau) + C(t, \tau). \tag{44}$$

The first term is the discount bond price with a face value of F. The second term is a call option on the bundle of the commodity with a strike price of F. The discount bond price in this model is given by (11). The option component can be readily derived from the futures option pricing formula by setting $\tilde{\tau} = \tau$, i.e., the maturity of the option and the maturity of the underlying futures contract are identical.

3. Conclusion

This paper proposes a new model to value commodity derivatives with stochastic convenience yields, stochastic interest rates, stochastic volatility and simultaneous jumps in the spot price and spot volatility. This model can capture many important character-istics of commodity returns. Two new features of the proposed model are stochastic volatility and simultaneous jumps in the spot price and spot volatility. They are employed to improve the pricing of commodity derivatives and options in particular. Closed-form valuation formulas for forwards, futures, futures options, geometric Asian options and commodity-linked bonds are obtained. I find that stochastic volatility and jumps do *not* alter the futures price at a given point in time. However, numerical examples show that they play important roles in pricing options on futures. Testing the proposed model using commodity futures and options data is left for future research.

Appendix 1: Futures Pricing Formula

The futures pricing formula is of the form in (13). The partial derivatives of H are: $H_L = H$, $H_{LL} = H$, $H_r = \beta_r H$, $H_{rr} = \beta_r^2 H$, $H_\delta = \beta_\delta H$, $H_{\delta\delta} = \beta_\delta^2 H$, $H_{L\delta} = \beta_\delta H$, $H_\tau = [\beta_0' + r\beta_r' + \delta\beta_\delta']H$. Substitute the above partial derivatives into (12). Grouping the resulting PDE by state variables r and δ, I obtain the following ordinary differential equations (ODEs):

$$\begin{aligned}
\beta_r' &= \tfrac{1}{2}\sigma_r^2\beta_r^2 - \kappa_r\beta_r + 1 \\
\beta_\delta' &= -1 - \kappa_\delta\beta_\delta \\
\beta_0' &= \tfrac{1}{2}\beta_\delta^2 + \sigma_S\sigma_\delta\rho_1\beta\delta + \theta_r\beta r + \theta_\delta\beta_\delta.
\end{aligned}$$

Solving the above ODEs subject to $\beta_r(0) = 0$, $\beta_\delta(0) = 0$ and $\beta_0(0) = 0$ gives (13).

Appendix 2: Futures Options Pricing Formula

$$\begin{aligned}
C(t,\tau) &= E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times \max(0, H(t+\tau, \tilde{\tau}-\tau) - K)\right\} \\
&= E^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right) \times H(t+\tau, \tilde{\tau}-\tau) | H > K\right\} \\
&\quad - KE^Q\left\{\exp\left(-\int_t^{t+\tau} r(u)du\right)\middle| H > K\right\} \\
&= G(t,\tau)E^Q\left\{\frac{\exp\left(-\int_t^{t+\tau} r(u)du\right)}{G(t,\tau)} \times H(t+\tau, \tilde{\tau}-\tau) | H > K\right\} \\
&\quad - KE^Q\left\{\frac{\exp\left(-\int_t^{t+\tau} r(u)\right)du}{B(t,\tau)}\middle| H > K\right\} \\
&= G(t,\tau)\Pi_1(t,\tau) - KB(t,\tau)\Pi_2(t,\tau).
\end{aligned}$$

Similar to those of futures, I obtain the following ODEs:

$$\begin{aligned}
\vartheta_r' &= \tfrac{1}{2}\sigma_r^2\vartheta_r^2 + \left[\sigma_r^2 i\phi\beta_r - \kappa_r\right]\vartheta_r + i\phi - 1 - \tfrac{1}{2}\sigma_r^2\phi^2\beta_r^2 - \kappa_r i\phi\beta_r \\
\vartheta_\delta' &= -i\phi - i\phi\beta_\delta\kappa_\delta - \kappa_\delta\vartheta_\delta \\
\vartheta_V' &= \tfrac{1}{2}\sigma_V^2\vartheta_V^2 + (\sigma_V\rho_2 i\phi - \kappa_V)\vartheta_V - \tfrac{1}{2}i\phi - \tfrac{1}{2}\phi^2 \\
\vartheta_0' &= \tfrac{1}{2}\sigma_\delta^2\vartheta_\delta^2 + \sigma_\delta^2 i\phi\beta_\delta\vartheta_\delta + \sigma_S\sigma\delta\rho_1 i\phi\vartheta_\delta + \theta_\delta\vartheta_\delta + \theta_r\vartheta_r + \theta_V\vartheta_V \\
&\quad + \frac{\lambda e^{i\phi(ln(1+\mu_J)-0.5\sigma_J^2)-0.5\phi^2\sigma_J^2}}{1-\theta\vartheta_V} - \lambda - \tfrac{1}{2}\sigma_S^2\phi^2 - \tfrac{1}{2}\sigma_S^2\phi^2\beta_\delta^2 \\
&\quad - \phi^2\beta_\delta\sigma_S\sigma_\delta\rho_1 - \lambda\mu_J i\phi - \tfrac{1}{2}\sigma_S^2 i\phi + \theta_r i\phi\beta_r + \theta_\delta i\phi\beta_\delta.
\end{aligned}$$

Solving the above ODEs subject to $\vartheta_r(0) = 0,\ \vartheta_V(0) = 0,\ \vartheta_\delta(0) = 0,\ \vartheta_0(0) = 0$ yields the formula

$$\begin{aligned}
&\vartheta_0(\tau) \\
&= -\frac{\theta_V}{\sigma_V^2}\left[2\ln\left(1 - \frac{(\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi)\left(1 - e^{-\xi_V^*\tau}\right)}{2\xi_V^*}\right)\right] \\
&\quad - \frac{\theta_r}{\sigma_r^2}\left[2\ln\left(1 - \frac{(\xi_r^* - \kappa_r + \sigma_r^2 i\phi\beta_r)\left(1 - e^{-\xi_r^*\tau}\right)}{2\xi_r^*}\right)\right] \\
&\quad - \frac{\theta_V}{\sigma_V^2}\,[\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi]\tau - \frac{\theta_r}{\sigma_r^2}\,[\xi_r^* - \kappa_r + \sigma_r^2 i\phi\beta_r]\tau \\
&\quad - \left(\lambda + \tfrac{1}{2}\sigma_S^2\phi^2 + \tfrac{1}{2}\sigma_\delta^2\phi^2\beta_\delta^2 + \phi^2\beta_\delta\sigma_S\sigma_\delta\rho_1 + \lambda\mu_J i\phi + \tfrac{1}{2}\sigma_S^2 i\phi - \theta_r i\phi\beta_r - \theta_\delta i\phi\beta_\delta\right)\tau \\
&\quad + \left(\frac{-\sigma_\delta^2\phi^2(1+\beta_\delta)^2}{2\kappa_\delta^2} - \frac{(i\phi + i\phi\beta_\delta)(\sigma_\delta^2 i\phi\beta_\delta + \sigma_S\sigma_\delta\rho_1 i\phi + \theta_\delta)}{\kappa_\delta}\right)\tau \\
&\quad + \frac{(-i\phi - i\phi\beta_\delta)(\sigma_\delta^2 i\phi\beta_\delta + \sigma_S\sigma_\delta\rho_1 i\phi + \theta_\delta)}{\kappa_\delta^2}\,e^{-\kappa_\delta\tau} + \frac{\sigma_\delta^2(i\phi + i\phi\beta_\delta)^2}{-4\kappa_\delta^3}\,e^{-2\kappa_\delta\tau} \\
&\quad + \frac{\sigma_\delta^2(i\phi + i\phi\beta_\delta)^2}{\kappa_\delta^3}\,e^{-\kappa_\delta\tau} \\
&\quad + \frac{3\sigma_\delta^2(i\phi + i\phi\beta_\delta)^2 - 4\kappa_\delta(i\phi + i\phi\beta_\delta)(\sigma_\delta^2 i\phi\beta_\delta + \sigma_S\sigma_\delta\rho_1 i\phi + \theta_\delta)}{-4\kappa_\delta^3} \\
&\quad + \lambda M(\xi_V^* + \kappa_V - \sigma_V\rho_2 i\phi)\frac{\tau}{\xi_V^* + \kappa_V - \sigma_V\rho_2 i\phi - \theta i\phi(i\phi - 1)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\ln(2\xi_V^* - [\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi + \theta i\phi(i\phi-1)](1-e^{-\xi_V^*\tau}))}{-\xi_V^*(\xi_V^* + \kappa_V - \sigma_V\rho_2 i\phi - \theta i\phi(i\phi-1))} \\
&+\lambda M(\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi)\frac{1}{-\xi_V^*[\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi + \theta i\phi(i\phi-1)]} \\
&\times \ln\big(2\xi_V^* - [\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi + \theta i\phi(i\phi-1)]\big(1-e^{-\xi_V^*\tau}\big)\big) \\
&+\ln(2\xi_V^*)\lambda M\left(\frac{\xi_V^* + \kappa_V - \sigma_V\rho_2 i\phi}{-\xi_V^*(\xi_V^* + \kappa_V - \sigma_V\rho_2 i\phi - \theta i\phi(i\phi-1))}\right. \\
&\qquad\left. - \frac{\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi}{-\xi_V^*[\xi_V^* - \kappa_V + \sigma_V\rho_2 i\phi + \theta i\phi(i\phi-1)]}\right)
\end{aligned}
$$

where

$$
M = e^{i\phi\left(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2\right) - \frac{1}{2}\phi^2\sigma_J^2}.
$$

Appendix 3: Asian Option Pricing Formula

$$
\begin{aligned}
\gamma_0(\tau) = &-\frac{\theta_V}{\sigma_V^2}\left[2\ln\left(1 - \frac{(\eta_V - \kappa_V + \sigma_V\rho_2\tau i\phi)(1-e^{-\eta_V\tau})}{2\eta_V^*}\right)\right] \\
&-\frac{\theta_r}{\sigma_r^2}\left[2\ln\left(1 - \frac{(\eta_r - \kappa_r)(1-e^{-\eta_r\tau})}{2\eta_r}\right)\right] \\
&-\frac{\theta_V}{\sigma_V^2}[\eta_V - \kappa_V + \sigma_V\rho_2\tau i\phi]\tau - \frac{\theta_r}{\sigma_r^2}[\eta_r - \kappa_r]\tau \\
&+\left(-\tfrac{1}{2}\sigma_S^2\phi^2\tau^2 - \lambda\mu_J i\phi\tau + i\phi\beta_0 - \lambda\right)\tau \\
&+\left(\frac{(i\phi\beta_\delta - i\phi\tau)(\sigma_S\sigma_\delta\rho_1 i\phi\tau + \theta_\delta)}{-\kappa_\delta} + \frac{\sigma_\delta^2(i\phi\beta_\delta - i\phi\tau)^2}{\kappa_\delta^2}\right)\tau \\
&+\frac{(-i\phi\beta_\delta + i\phi\tau)(\sigma_S\sigma\delta\rho_1 i\phi\tau + \theta_\delta)}{\kappa_\delta^2}e^{-\kappa_\delta\tau} + \frac{\sigma_\delta^2(i\phi\beta_\delta - i\phi\tau)^2}{-4\kappa_\delta^3}e^{-2\kappa_\delta\tau} \\
&-\frac{\sigma_\delta^2(i\phi\beta_\delta - i\phi\tau)^2}{-\kappa_\delta^3}e^{-\kappa_\delta\tau} + \frac{3\sigma_\delta^2(i\phi\beta_\delta - i\phi\tau)^2 + 2\kappa_\delta(i\phi\beta_\delta - i\phi\tau)(\sigma_\delta\sigma_S\rho_1 i\phi\tau + \theta_\delta)}{-4\kappa_\delta^3}
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda M(\eta_V + \kappa_V - \sigma_V \rho_2 i\phi) \frac{\tau}{\eta_V + \kappa_V - \sigma_V \rho_2 i\phi - \theta(i\phi\tau + \phi^2\tau^2)} \\
&- \frac{\ln\left(2\eta_V - \left(\eta_V - \kappa_V + \sigma_V \rho_2 i\phi + \theta i\phi\tau + \theta\phi^2\tau^2\right)(1 - e^{-\eta_V \tau})\right)}{(\eta_V + \kappa_V - \sigma_V \rho_2 i\phi - \theta(i\phi\tau + \phi^2\tau^2))\eta_V} \\
&+ \lambda M \frac{\eta_V - \kappa_V + \sigma_V \rho_2 i\phi}{(\eta_V - \kappa_V + \sigma_V \rho_2 i\phi + \theta i\phi\tau + \theta\phi^2\tau^2)\eta_V} \\
&\times \ln(2\eta_V - (\eta_V - \kappa_V + \sigma_V \rho_2 i\phi + \theta i\phi\tau + \theta\phi^2\tau^2)(1 - e^{-\eta_V \tau})) + \lambda M \\
&\times \ln(2\eta_V) \left(\frac{\eta_V + \kappa_V - \sigma_V \rho_2 i\phi}{(\eta_V + \kappa_V - \sigma_V \rho_2 i\phi - \theta(i\phi\tau + \phi^2\tau^2))\eta_V} \right. \\
&\qquad \left. - \frac{\eta_V - \kappa_V + \sigma_V \rho_2 i\phi}{(\eta_V - \kappa_V + \sigma_V \rho_2 i\phi + \theta i\phi\tau + \theta\phi^2\tau^2)\eta_V} \right)
\end{aligned}
$$

where

$$M = e^{i\phi\left(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2\right) - \frac{1}{2}\phi^2\sigma_J^2}.$$

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Notes

1. See Schwartz (1982), Brennan and Schwartz (1985) and Gibson and Schwartz (1991).
2. See also Bakshi, Cao and Chen (2000) and Eraker, Johannes and Polson (2002).
3. Most exchange-traded commodity options are options on futures rather than options on spots because futures markets are more established and standardized.
4. In practice, one needs to value the integration numerically. I call this formula a closed-form formula in the same sense as the Black-Scholes formula, which involves numerically computing normal probabilities.
5. Specifically, the total futures return volatility under Model 1 is given by (15): $\sigma_H^2(\tau) = \sigma_S^2 + V + \lambda[\mu_J^2 + (e^{\sigma_J^2} - 1)(1 + \mu_J)^2] + \beta_r^2 r\sigma_r^2 + \beta_\delta^2\sigma_\delta^2 + \rho_1\beta_\delta\sigma_S\sigma_\delta$. Take Model 2 as an example. Recall that Model 2 does not allow for jumps, but allows for stochastic convenience yields, stochastic interest rates and stochastic volatility. The value of V that keeps the total volatility the same as Model 1 would be V (of model 1) $+\lambda[\mu_J^2 + (e^{\sigma_J^2} - 1)(1 + \mu_J)^2]$, which is 0.0425. Correspondingly, θ_V, the parameter that determines the long run mean of V, is now 0.085, or 2 $(\kappa_V) \times 0.0425$.
6. I would like to thank Charles Cao for suggesting doing this.

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