

A Variational Multiscale Finite Element Method for Transient Solid Mechanics

Xianyi Zeng
xzeng@utep.edu

Department of Mathematical Sciences
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Outline

- 1 Introduction and Motivation
- 2 The Variational Multiscale Approach: Overview
- 3 The Variational Multiscale Approach for Solid Mechanics
- 4 Further Applications and Extensions of the VMS-FEM

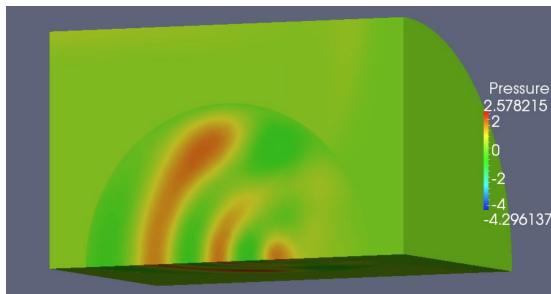


Fluid-Structure Interaction with Complex Geometries

Research target

- A general fluid-structure interaction computation framework.
 - Complicated geometries.
 - Interaction between strong discontinuities and material interfaces.
- Compressible fluids.
- Solids with general material properties.

Shock-gel interaction

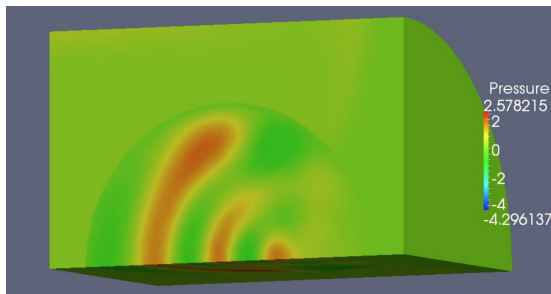


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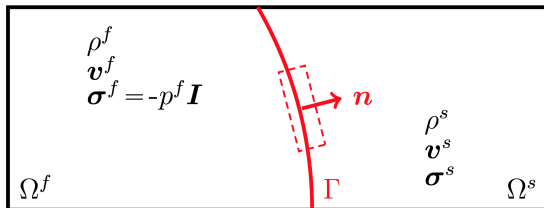


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A general FSI problem



$$(v^f - v^s) \cdot n = 0$$

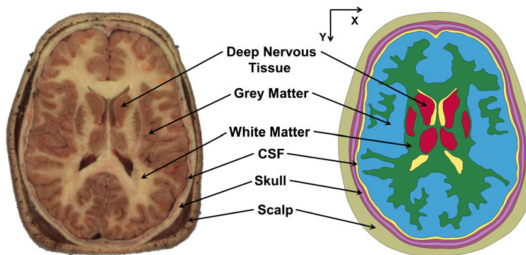
$$(\sigma^f - \sigma^s)n = 0 \Rightarrow -p^f n = \sigma^s n$$

Stabilized FEM Using Low-Order Simplicial Elements

Motivation I: meshing complex objects

- Realistic objects are often complex.
- Meshing with quadrilateral/hexahedral elements is time-consuming.
- Meshing with simplicial elements is highly automated.

2D section of the human brain¹



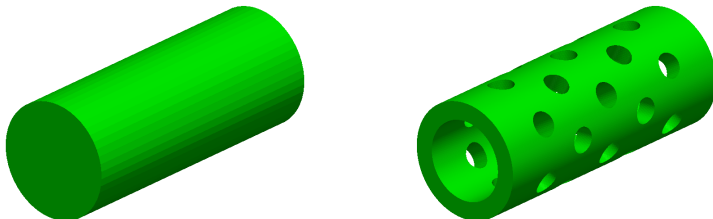
¹Matthew B. Panzer et al. *Ann. Biomed. Eng.*, 2012.

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Meshing with different elements

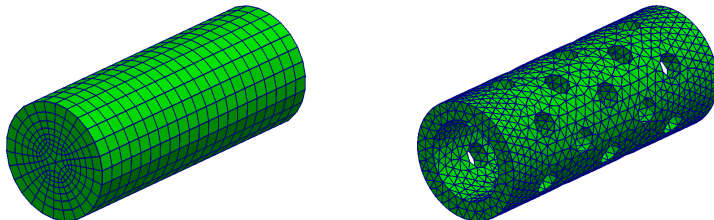


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Meshing with different elements



Stabilized FEM Using Low-Order Simplicial Elements

$$\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d$$

$$\boldsymbol{\sigma} = \mathbf{C} \nabla \mathbf{u} = p \mathbf{I} + \boldsymbol{\sigma}^{dev}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b$$

Motivation II: standard Galerkin formulations (static case)

- Optimal for compressible materials.
- Variational formulation:

Find $\mathbf{u} \in \mathcal{S}$ such that for all $\boldsymbol{\phi} \in \mathcal{V}$:

$$-(\nabla \boldsymbol{\phi}, \mathbf{C} \nabla \mathbf{u}) + (\boldsymbol{\phi}, \rho_0 \mathbf{b}) = 0$$

- For compressible materials, all components of $\mathbf{C}$ have the similar magnitudes.



Stabilized FEM Using Low-Order Simplicial Elements

$$\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d$$

$$\boldsymbol{\sigma} = \mathbf{C} \nabla \mathbf{u} = p \mathbf{I} + \boldsymbol{\sigma}^{dev}$$

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Motivation II: standard Galerkin formulations (static case)

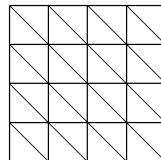
- There are issues with incompressible (nearly incompressible) materials.
- p serves as the Lagrangian multiplier for $\nabla \cdot \mathbf{u} = 0$.
- $\mathcal{P}_1/\mathcal{P}_0$ formulation: too many constraints $\Rightarrow$ volume locking.

- Unknowns: $N_{node} \times d$.
- Constraints: N_{elem} .
- Example: triangular meshes, $d = 2$,

$$N_{elem}/N_{node} \approx 2.$$

- Example: tetrahedral meshes, $d = 3$,

$$N_{elem}/N_{node} \approx 4 \sim 6.$$



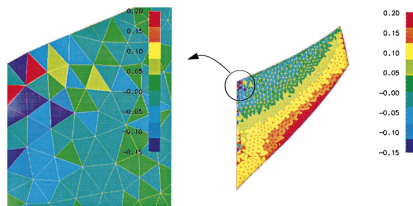
Stabilized FEM Using Low-Order Simplicial Elements

$$\begin{aligned}\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} &= 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d \\ \boldsymbol{\sigma} &= \mathbf{C} \nabla \mathbf{u} = p \mathbf{I} + \boldsymbol{\sigma}^{dev} \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{u}_b\end{aligned}$$

Motivation II: standard Galerkin formulations (static case)

- There are issues with incompressible (nearly incompressible) materials.
- p serves as the Lagrangian multiplier for $\nabla \cdot \mathbf{u} = 0$.
- $\mathcal{P}_1/\mathcal{P}_1$ formulation: instability

Cook's membrane problem: first principal stress¹



¹Ottmar Klaas et al. *Comput. Methods Appl. Mech. Eng.*, 1999.

Stabilized FEM Using Low-Order Simplicial Elements

$$\begin{aligned}\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} &= 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d \\ \boldsymbol{\sigma} &= \mathbf{C} \nabla \mathbf{u} = p \mathbf{I} + \boldsymbol{\sigma}^{dev} \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{u}_b\end{aligned}$$

Motivation II: standard Galerkin formulations (static case)

- There are issues with incompressible (nearly incompressible) materials.
- p serves as the Lagrangian multiplier for $\nabla \cdot \mathbf{u} = 0$.
- Existing solutions in literature:
 - More sophisticated elements.
 - $\mathcal{P}_2/\mathcal{P}_1$ formulation.
 - Triangular/tetrahedral composite elements.
 - Stabilized formulation: add terms to satisfy the LBB (*inf-sup*) condition.
 - Bubble functions.
 - The variational multi-scale (VMS) approach.
 - Rewrite the equations in first-order form.
 - Strain/velocity formulation.
 - Displacement gradient formulation.



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The VMS-Approach: Overview

The variational multiscale approach for a linear problem

$$\begin{aligned}\mathcal{L}u &= f \quad \text{on } \Omega \\ u|_{\partial\Omega} &= u_g\end{aligned}$$

Suppose sufficient smoothness

- Variational problem: seek $u \in \mathcal{S}$ such that for all $\phi \in \mathcal{V}$

$$(\phi, \mathcal{L}u) = (\phi, f).$$

- Standard Galerkin method: seek $u^h \in \mathcal{S}^h \subset \mathcal{S}$ such that for all $\phi^h \in \mathcal{V}^h \subset \mathcal{V}$

$$(\phi^h, \mathcal{L}u^h) = (\phi^h, f).$$

- The VMS-approach: decomposition of coarse scales and fine (subgrid) scales

$$\mathcal{S} = \mathcal{S}^h \oplus \mathcal{S}', \quad \mathcal{V} = \mathcal{V}^h \oplus \mathcal{V}';$$

and

$$u = u^h + u', \quad \phi = \phi^h + \phi'.$$

- Assume $\mathcal{S}' = \mathcal{V}'$: $\phi'|_{\partial\Omega} = u'|_{\partial\Omega} = 0$.



The VMS-Approach: Overview

Coarse scale problem

- Test against ϕ^h :

$$\begin{aligned}
 (\phi^h, \mathcal{L}u^h + \mathcal{L}u') &= (\phi^h, f) \\
 \iff (\phi^h, \mathcal{L}u^h) + (\mathcal{L}^*\phi^h, u') &= (\phi^h, f).
 \end{aligned}$$

- Need to model u' .
- The stabilized method:

$$u' = -\tau(\mathcal{L}u^h - f).$$

Fine scale problem

- Test against ϕ' :

$$\begin{aligned}
 (\phi', \mathcal{L}u^h) + (\phi', \mathcal{L}u') &= (\phi', f) \\
 \iff (\phi', \mathcal{L}u') &= -(\phi', \mathcal{L}u^h - f).
 \end{aligned}$$

- u' solves a global problem in $\mathcal{S}'$.



The VMS-Approach: Localization and Interpretation

Localization assumption

- Let $\mathcal{T} = \{\Omega_e\}$ be a tessellation of Ω .
- Computing L_2 -projection of $-(\mathcal{L}u^h - f)$ onto $\mathcal{S}'$ needs global solution.
- **Localization assumption:** $\phi'|_{\partial\Omega_e} = u'|_{\partial\Omega_e} = 0$.
- $u'|_{\Omega_e}$ solves the local homogeneous Dirichlet problem:

$$\begin{aligned}\mathcal{L}u' &= -(\mathcal{L}u^h - f) \quad \text{on} \quad \Omega_e, \\ u'|_{\partial\Omega_e} &= 0.\end{aligned}$$



The VMS-Approach: Localization and Interpretation

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Stabilization parameter

- Exact u' computed by using Green's function of the problem:

$$u'(\mathbf{y}) = - \int_{\Omega_e} G(\mathbf{x}, \mathbf{y})(\mathcal{L}u^h - f)(\mathbf{x}) d\Omega_e,$$

- In a stabilized method, $u' \stackrel{\text{def}}{=} -\tau(\mathcal{L}u^h - f)$: suggesting the approximation:

$$\tau \approx \frac{1}{\mu(\Omega_e)} \int_{\Omega_e} \int_{\Omega_e} G(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}.$$



The VMS-Approach: Localization and Interpretation

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Stabilization parameter

- τ typically scales with the time:
 - 1D advection-diffusion equation ($\mathcal{L} = a \frac{d}{dx} - \kappa \frac{d^2}{dx^2}$):

$$\tau = \frac{h}{2a} \left(\coth \frac{ah}{2\kappa} - \frac{2\kappa}{ah} \right).$$
- Extension to transient problems ($\mathcal{L} \rightarrow \partial_t + \mathcal{L}$), $\tau \sim \Delta t$.



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- ① Introduction and Motivation
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- ③ **The Variational Multiscale Approach for Solid Mechanics**
 - The VMS-Approach for Isotropic Linear Elasticity
 - The VMS-Approach for Neo-Hookean Solids
- ④ Further Applications and Extensions of the VMS-FEM



The VMS-Approach: Solid Mechanics with Infinitesimal Strain

Governing equations in $\mathbb{R}^3$

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{v}, \\ \rho_0 \dot{\mathbf{v}} &= \nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b}, \\ \boldsymbol{\sigma} &= \mathbf{C} \nabla \mathbf{u}, \quad \mathbf{C}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).\end{aligned}$$

Stabilization for the static problem by Hughes, Franca, and Balestra¹

- Displacement/pressure formulation.
- Enrich the displacement space with the fine-scale quantities:

$$\mathbf{u}' = -\tau (\nabla p^h + \nabla \cdot \boldsymbol{\sigma}^{dev} + \rho_0 \mathbf{b}), \quad \tau = c_\tau \frac{h_e^2}{2\mu}.$$

- Extension to transient problems is not straightforward:
 - Elliptic vs. hyperbolic.
 - No displacement residual in transient problems.
 - Cannot apply the “static stabilization” directly.



¹T. J. R. Hughes, L. P. Franca, M. Balestra, *Comput. Methods Appl. Mech. Eng.*, 1986.

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Rate equation for pressure

- Isochoric decomposition of the Cauchy stress $\boldsymbol{\sigma} = p\mathbf{I} + \boldsymbol{\sigma}^{dev}$:

$$p\mathbf{I} = \mathbf{C}^{vol} \nabla \mathbf{u}, \quad \mathbf{C}^{vol} \propto \lambda + 2\mu/3 = \frac{E}{3(1-2\nu)}$$

$$\boldsymbol{\sigma}^{dev} = \mathbf{C}^{dev} \nabla \mathbf{u}, \quad \mathbf{C}^{dev} \propto 2\mu/3 = 2G/3.$$

- Solve the rate equation for pressure:

$$\begin{aligned}\rho_0 \dot{\mathbf{v}} &= \nabla p + \nabla \cdot \boldsymbol{\sigma}^{dev} + \rho_0 \mathbf{b}, \\ \dot{p} &= \kappa \nabla \cdot \mathbf{v}, \quad \kappa = \lambda + 2\mu/3,\end{aligned}$$

$$\boldsymbol{\sigma}^{dev} = 2\mu \tilde{\nabla}^s \mathbf{u}, \quad \tilde{\nabla}^s \mathbf{u} \stackrel{\text{def}}{=} 2^{-1} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - 3^{-1} \nabla \cdot \mathbf{u} \mathbf{I}.$$

- Use velocity residual to define $\mathbf{v}'$ to stabilize the pressure equation.



The VMS-Approach: Solid Mechanics with Infinitesimal Strain

The variational formulation of the VMS-FEM

- Assume homogeneous displacement boundary condition.
- The spaces:

$$\mathcal{S}_v^h = \mathcal{V}_v^h = \{\psi^h \in (C^0(\Omega))^3 : \psi^h|_{\Omega_e} \in (\mathcal{P}_1(\Omega_e))^3, \psi^h|_{\partial\Omega} = 0\}$$

$$\mathcal{S}_s^h = \mathcal{V}_s^h = \{\phi^h \in C^0(\Omega) : \phi^h|_{\Omega_e} \in \mathcal{P}_1(\Omega_e)\}$$

- The variational problem: find $\mathbf{u}^h, \mathbf{v}^h \in \mathcal{S}_v^h$ and $p^h \in \mathcal{S}_s^h$, such that

$$0 = (\psi^h, \dot{\mathbf{u}}^h - \mathbf{v}^h - \tau \dot{\mathbf{v}}^h),$$

$$0 = (\psi^h, \rho_0 \dot{\mathbf{v}}^h) + (\nabla \cdot \psi^h, p^h + p') + (\nabla \psi^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h) - (\psi^h, \rho_0 \mathbf{b}),$$

$$0 = (\phi^h, \kappa^{-1} \dot{p}^h - \nabla \cdot \mathbf{v}^h) + (\nabla \phi^h, \mathbf{v}').$$

for all $\psi^h \in \mathcal{V}_v^h$ and $\phi^h \in \mathcal{V}_s^h$.

- Fine scale variables ($\tau = \Delta t/2$):

$$p' = -\tau(\dot{p}^h - \kappa \nabla \cdot \mathbf{v}^h),$$

$$\mathbf{v}' = -\frac{\tau}{\rho_0} \left(\rho_0 \dot{\mathbf{v}}^h - \nabla p^h - \nabla \cdot \left(2\mu \tilde{\nabla}^s \mathbf{u}^h \right) - \rho_0 \mathbf{b} \right).$$



Energy Stability

$$0 = (\boldsymbol{\psi}^h, \dot{\mathbf{u}}^h - \mathbf{v}^h - \tau \dot{\mathbf{v}}^h), \quad (1)$$

$$0 = (\boldsymbol{\psi}^h, \rho_0 \dot{\mathbf{v}}^h) + (\nabla \cdot \boldsymbol{\psi}^h, p^h - \tau(\dot{p}^h - \kappa \nabla \cdot \mathbf{v}^h)) + (\tilde{\nabla}^s \boldsymbol{\psi}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h), \quad (2)$$

$$0 = (\phi^h, \kappa^{-1} \dot{p}^h - \nabla \cdot \mathbf{v}^h) + (\nabla \phi^h, -\tau \rho_0^{-1} (\rho_0 \dot{\mathbf{v}}^h - \nabla p^h)). \quad (3)$$

An energy stability proof of the VMS-FEM

- Equivalent to (1): nodal update for displacement:

$$\frac{d}{dt}(\tilde{\nabla}^s \mathbf{u}^h) = \tilde{\nabla}^s \dot{\mathbf{u}}^h = \tilde{\nabla}^s \mathbf{v}^h + \tau \tilde{\nabla}^s \dot{\mathbf{v}}^h.$$



Energy Stability

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- Test (2) and (3) against $\boldsymbol{\psi}^h = \mathbf{v}^h$ and $\phi^h = p^h$, respectively:

$$0 = (\mathbf{v}^h, \rho_0 \dot{\mathbf{v}}^h) + (\nabla \cdot \mathbf{v}^h, p^h) - (\nabla \cdot \mathbf{v}^h, \tau \dot{p}^h) + (\nabla \cdot \mathbf{v}^h, \tau \kappa \nabla \cdot \mathbf{v}^h) + (\tilde{\nabla}^s \mathbf{v}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h),$$

$$0 = (p^h, \kappa^{-1} \dot{p}^h - \nabla \cdot \mathbf{v}^h) - (\nabla p^h, \tau \dot{\mathbf{v}}^h) + (\nabla p^h, \tau \rho_0^{-1} \nabla p^h).$$



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- Test (2) and (3) against $\boldsymbol{\psi}^h = \tau \dot{\mathbf{v}}^h$ and $\phi^h = \tau \dot{p}^h$, respectively:

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$$(\tau \tilde{\nabla}^s \dot{\mathbf{v}}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h),$$

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- Equivalent to (1): nodal update for displacement:

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- Test (2) and (3) against $\boldsymbol{\psi}^h = \tau \dot{\mathbf{v}}^h$ and $\phi^h = \tau \dot{p}^h$, respectively:

$$0 = (\tau \dot{\mathbf{v}}^h, \rho_0 \dot{\mathbf{v}}^h) + (\tau \nabla \cdot \dot{\mathbf{v}}^h, p^h) - (\tau \nabla \cdot \dot{\mathbf{v}}^h, \tau \dot{p}^h) + (\tau \nabla \cdot \dot{\mathbf{v}}^h, \tau \kappa \nabla \cdot \mathbf{v}^h) +$$

$$(\tilde{\nabla}^s \dot{\mathbf{u}}^h - \tilde{\nabla}^s \mathbf{v}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h),$$

$$0 = (\tau \dot{p}^h, \kappa^{-1} \dot{p}^h - \nabla \cdot \mathbf{v}^h) - (\tau \nabla \dot{p}^h, \tau \dot{\mathbf{v}}^h) + (\tau \nabla \dot{p}^h, \tau \rho_0^{-1} \nabla p^h).$$



Energy Stability

$$\begin{aligned}
 0 &= \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \rho_0^{1/2} \mathbf{v}^h \right\|^2}_{\text{A-1}} + \underbrace{(\nabla \cdot \mathbf{v}^h, p^h)}_{\text{A-2}} - \underbrace{(\nabla \cdot \mathbf{v}^h, \tau \dot{p}^h)}_{\text{A-3}} + \underbrace{\left\| (\tau \kappa)^{1/2} \nabla \cdot \mathbf{v}^h \right\|^2}_{\text{A-4}} + \\
 &\quad \underbrace{\frac{1}{2} \frac{d}{dt} \left\| (2\mu)^{1/2} \tilde{\nabla}^s \mathbf{u}^h \right\|_{\text{F}}^2}_{\text{A-5}} - \underbrace{(\tau \tilde{\nabla}^s \dot{\mathbf{v}}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h)}_{\text{A-6}}, \\
 0 &= \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \kappa^{-1/2} p^h \right\|^2}_{\text{B-1}} - \underbrace{(p^h, \nabla \cdot \mathbf{v}^h)}_{\text{B-2}} - \underbrace{(\nabla p^h, \tau \dot{\mathbf{v}}^h)}_{\text{B-3}} + \underbrace{\left\| (\tau / \rho_0)^{1/2} \nabla p^h \right\|^2}_{\text{B-4}}, \\
 0 &= \underbrace{\left\| (\tau \rho_0)^{1/2} \dot{\mathbf{v}}^h \right\|^2}_{\text{C-1}} + \underbrace{(\tau \nabla \cdot \dot{\mathbf{v}}^h, p^h)}_{\text{C-2}} + \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \tau \kappa^{1/2} \nabla \cdot \mathbf{v}^h \right\|^2}_{\text{C-3}} - \underbrace{(\tau \nabla \cdot \dot{\mathbf{v}}^h, \tau \dot{p}^h)}_{\text{C-4}} + \\
 &\quad \underbrace{(\tau \tilde{\nabla}^s \dot{\mathbf{v}}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h)}_{\text{C-5}}, \\
 0 &= \underbrace{\left\| (\tau / \kappa)^{1/2} \dot{p}^h \right\|^2}_{\text{D-1}} - \underbrace{(\tau \dot{p}^h, \nabla \cdot \mathbf{v}^h)}_{\text{D-2}} - \underbrace{(\tau \nabla \dot{p}^h, \tau \dot{\mathbf{v}}^h)}_{\text{D-3}} + \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \tau \rho_0^{-1/2} \nabla p^h \right\|^2}_{\text{D-4}}.
 \end{aligned}$$



Energy Stability

$$0 = \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \rho_0^{1/2} \mathbf{v}^h \right\|^2}_{\text{A-1}} + \underbrace{\cancel{(\nabla \cdot \mathbf{v}^h, p^h)}}_{\text{A-2}} - \underbrace{(\nabla \cdot \mathbf{v}^h, \tau \dot{p}^h)}_{\text{A-3}} + \underbrace{\left\| (\tau \kappa)^{1/2} \nabla \cdot \mathbf{v}^h \right\|^2}_{\text{A-4}} +$$

$$\underbrace{\frac{1}{2} \frac{d}{dt} \left\| (2\mu)^{1/2} \tilde{\nabla}^s \mathbf{u}^h \right\|_F^2}_{\text{A-5}} - \underbrace{\cancel{(\tau \tilde{\nabla}^s \dot{\mathbf{v}}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h)}}_{\text{A-6}},$$

$$0 = \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \kappa^{-1/2} p^h \right\|^2}_{\text{B-1}} - \underbrace{\cancel{(p^h, \nabla \cdot \mathbf{v}^h)}}_{\text{B-2}} - \underbrace{(\nabla p^h, \tau \dot{\mathbf{v}}^h)}_{\text{B-3}} + \underbrace{\left\| (\tau / \rho_0)^{1/2} \nabla p^h \right\|^2}_{\text{B-4}},$$

$$0 = \underbrace{\left\| (\tau \rho_0)^{1/2} \dot{\mathbf{v}}^h \right\|^2}_{\text{C-1}} + \underbrace{(\tau \nabla \cdot \dot{\mathbf{v}}^h, p^h)}_{\text{C-2}} + \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \tau \kappa^{1/2} \nabla \cdot \mathbf{v}^h \right\|^2}_{\text{C-3}} - \underbrace{\cancel{(\tau \nabla \cdot \dot{\mathbf{v}}^h, \tau \dot{p}^h)}}_{\text{C-4}} +$$

$$\underbrace{\cancel{(\tau \tilde{\nabla}^s \dot{\mathbf{v}}^h, 2\mu \tilde{\nabla}^s \mathbf{u}^h)}}_{\text{C-5}},$$

$$0 = \underbrace{\left\| (\tau / \kappa)^{1/2} \dot{p}^h \right\|^2}_{\text{D-1}} - \underbrace{(\tau \dot{p}^h, \nabla \cdot \mathbf{v}^h)}_{\text{D-2}} - \underbrace{\cancel{(\tau \nabla \dot{p}^h, \tau \dot{\mathbf{v}}^h)}}_{\text{D-3}} + \underbrace{\frac{1}{2} \frac{d}{dt} \left\| \tau \rho_0^{-1/2} \nabla p^h \right\|^2}_{\text{D-4}}.$$



Energy Stability

An energy stability proof of the VMS-FEM

- Define the energy of the solutions:

$$\mathcal{E}_\tau(t) = \left\| \rho_0^{1/2} \mathbf{v}^h \right\|^2 + \left\| \kappa^{-1/2} p^h \right\|^2 + \left\| (2\mu)^{1/2} \tilde{\nabla}^s \mathbf{u}^h \right\|_F^2 + \left\| \tau \kappa^{1/2} \nabla \cdot \mathbf{v}^h \right\|^2 + \left\| \tau \rho_0^{-1/2} \nabla p^h \right\|^2.$$

- The energy stability:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\tau(t) = & - \left\| (\tau \kappa)^{1/2} \nabla \cdot \mathbf{v}^h - (\tau / \kappa)^{1/2} \dot{p}^h \right\|^2 \\ & - \left\| (\tau \rho_0)^{1/2} \dot{\mathbf{v}}^h - (\tau / \rho_0)^{1/2} \nabla p^h \right\|^2 \leq 0. \end{aligned}$$

- What if $\tau = 0$ (no stabilization)?

$$\frac{d}{dt} \left[\left\| \rho_0^{1/2} \mathbf{v}^h \right\|^2 + \left\| \kappa^{-1/2} p^h \right\|^2 + \left\| (2\mu)^{1/2} \tilde{\nabla}^s \mathbf{u}^h \right\|_F^2 \right] = 0.$$



Energy Stability

An energy stability proof of the VMS-FEM

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$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\tau(t) = & - \left\| (\tau \kappa)^{1/2} \nabla \cdot \mathbf{v}^h - (\tau / \kappa)^{1/2} \dot{p}^h \right\|^2 \\ & - \left\| (\tau \rho_0)^{1/2} \dot{\mathbf{v}}^h - (\tau / \rho_0)^{1/2} \nabla p^h \right\|^2 \leq 0. \end{aligned}$$

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Finalize the VMS-FEM

The Time-Integrator: $t^n \rightarrow t^{n+1} = t^n + \Delta t$

- Determining Δt from the Courant-Friedrichs-Lewy condition.
- Bulk velocity (maximum characteristic speed) $\sqrt{(\kappa + 4\mu/3)/\rho_0}$.
- Abstract form of the ODE system:

$$\dot{U} = \mathcal{R}(U).$$

- Predictor/multi-corrector: fixed-point iteration about $t^n + \Delta t/2$.

$$U^{(0)} = U^n,$$

$$U^{(k+1)} = U^n + \Delta t \mathcal{R}((U^n + U^{(k)})/2), \quad k = 0, \dots, K-1,$$

$$U^{n+1} = U^{(K)}.$$

- $K \geq 2$ ensures second-order accuracy in time.
- $K \rightarrow \infty$ leads to the mid-point rule.
- The energy proof carries to the fully discretized method.



Numerical Examples: Accuracy Test

Swinging Plate

- 2D plate with plane strain.
- Exact solution on $\Omega = [0, 2]^2$:

$$\mathbf{u}(\mathbf{x}, t) = U_0 \sin \sqrt{\frac{\mu \pi^2}{2 \rho_0}} t \begin{bmatrix} -\sin \frac{\pi x}{2} \cos \frac{\pi y}{2} \\ \cos \frac{\pi x}{2} \sin \frac{\pi y}{2} \end{bmatrix},$$

$$\mathbf{v}(\mathbf{x}, t) = \partial_t \mathbf{u}(\mathbf{x}, t),$$

$$p(\mathbf{x}, t) = 0.$$

- Parameters: U_0, E, ρ_0, ν .

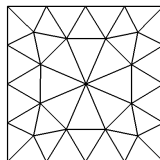
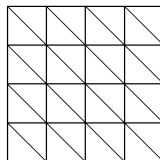
$$U_0 = 0.01\text{m},$$

$$E = 1.7 \times 10^7 \text{Pa},$$

$$\rho_0 = 1.1 \times 10^3 \text{kg/m}^3.$$

- Tests: $\nu = 0.45, 0.49995$.

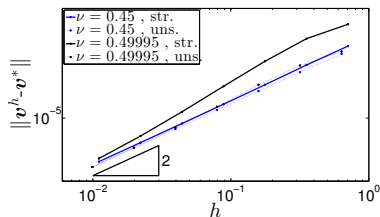
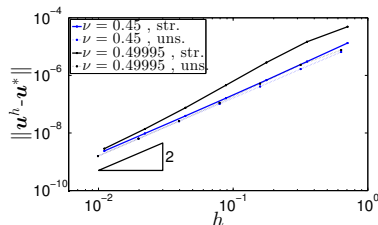
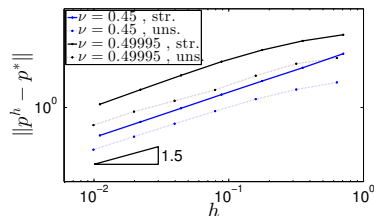
- Meshes:



Numerical Examples: Accuracy Test

Swinging Plate

- L_2 -convergence in p , u , and v .

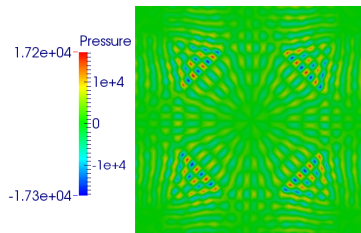


Numerical Examples: Accuracy Test

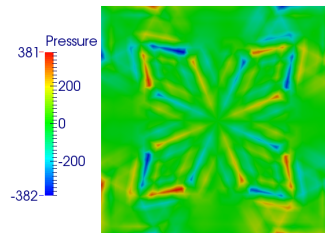
Swinging Plate

- Comparison with the “static stabilization¹”:

Static stabilization



VMS-stabilization



¹T. J. R. Hughes, L. P. Franca, M. Balestra, *Comput. Methods Appl. Mech. Eng.*, 1986.

Numerical Examples: Stability Test

Cook's membrane

- 2D membrane with plane strain.
- Nondimensionalized parameters:

$$\rho_0 = 1.0,$$

$$E = 250.0,$$

$$\nu = 0.49995.$$

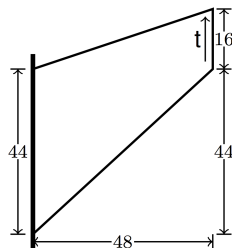
- Methods:

- Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_0$. (Fail!)
- Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_1$.
- VMS-FEM: $\mathcal{P}_1/\mathcal{P}_1$.

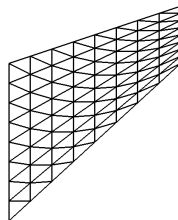
- Meshes:

$$2 \times 8^2, 2 \times 16^2, 2 \times 32^2, 2 \times 64^2.$$

Configuration.



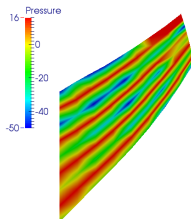
Mesh.



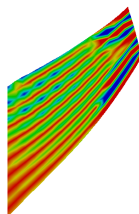
Numerical Examples: Stability Test

Cook's membrane

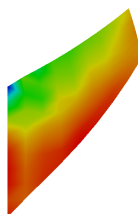
Galerkin 2×8^2



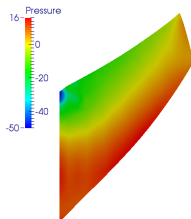
Galerkin 2×16^2



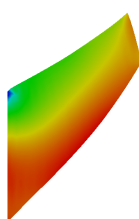
VMS-FEM 2×8^2



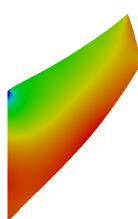
VMS-FEM 2×16^2



VMS-FEM 2×32^2



VMS-FEM 2×64^2



Numerical Examples: Stability Test

Cook's membrane



The VMS-Approach: Solid Mechanics with Finite Strain

Governing equations in $\mathbb{R}^3$

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{v}, \\ \rho J &= \rho_0, \\ \rho \dot{\mathbf{v}} &= \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \rho \mathbf{b}\end{aligned}$$

Problem variables

- Material coordinate $\mathbf{X}$ and current configuration $\mathbf{x}$:

$$\begin{aligned}\varphi: \Omega_0 &\rightarrow \Omega = \varphi(\Omega_0), \\ \mathbf{X} &\mapsto \mathbf{x} = \varphi(\mathbf{X}, t), \quad \forall \mathbf{X} \in \Omega_0, t \geq 0.\end{aligned}$$

- $\dot{}$ denotes time-derivative at fixed $\mathbf{X}$.
- Geometric quantities:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \nabla_{\mathbf{X}} \mathbf{x}, \quad J = \det \mathbf{F}.$$

- The right Cauchy-Green tensor and the second Piola-Kirchhoff tensor:

$$\mathbf{C} = \mathbf{F}^t \mathbf{F}, \quad \mathbf{S} = J \mathbf{F}^{-1} \boldsymbol{\sigma} (\mathbf{F}^{-1})^t.$$



The VMS-Approach: Solid Mechanics with Finite Strain

Governing equations in $\mathbb{R}^3$

$$\begin{aligned}\dot{\mathbf{u}} &= \mathbf{v}, \\ \rho J &= \rho_0, \\ \rho \dot{\mathbf{v}} &= \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \rho \mathbf{b}\end{aligned}$$

The hyperelastic constitutive model

- Stress-strain relation derived from stored-energy functions:

$$\mathbf{S} = 2\partial_{\mathbf{C}}\Psi(\mathbf{C}), \quad \Psi(\mathbf{C}) = U(J) + W(\overline{\mathbf{C}}), \quad \overline{\mathbf{C}} = J^{-2/3}\mathbf{C}.$$

- Extension of the linear model:

$$\begin{aligned}U(J) &= \frac{1}{4}\kappa(J^2 - 1) - \frac{1}{2}\kappa \ln J, \quad W(\overline{\mathbf{C}}) = \frac{1}{2}\mu(\text{tr } \overline{\mathbf{C}} - 3), \\ \Rightarrow \quad \begin{cases} \dot{\mathbf{p}} = U''(J)\dot{J} = \tilde{\kappa}(J)\nabla_{\mathbf{x}} \cdot \mathbf{v}, & \tilde{\kappa}(J) = \kappa(J + J^{-1})/2, \\ \boldsymbol{\sigma}^{dev} = J^{-5/3}\mathbf{F} \text{DEV}_{\mathbf{C}}[2\partial_{\overline{\mathbf{C}}}W]\mathbf{F}^t = \mu J^{-5/3} \text{dev}(\mathbf{F}\mathbf{F}^t). \end{cases} \end{aligned}$$

- $\text{DEV}_{\mathbf{C}}[\mathbf{A}] \stackrel{\text{def}}{=} \mathbf{A} - \frac{1}{3}(\mathbf{A} : \mathbf{C})\mathbf{C}^{-1} \Rightarrow \mathbf{F} \text{DEV}_{\mathbf{C}}[\bullet]\mathbf{F}^t = \text{dev}[\mathbf{F}(\bullet)\mathbf{F}^t].$



The VMS-Approach: Solid Mechanics with Finite Strain

The variational formulation of the VMS-FEM

- Idea of VMS for nonlinear operators: linearizing $\mathcal{L}(U^h + U')$ about U^h .
- The variational problem: find $\mathbf{u}^h, \mathbf{v}^h \in \mathcal{S}_v^h$ and $p^h \in \mathcal{S}_s^h$, such that

$$0 = (\boldsymbol{\psi}^h, \dot{\mathbf{u}}^h - \mathbf{v}^h - \tau \dot{\mathbf{v}}^h)_{\Omega},$$

$$0 = (\phi^h, \rho^h J - \rho_0)_{\Omega},$$

$$0 = (\boldsymbol{\psi}^h, \rho \dot{\mathbf{v}}^h)_{\Omega} + (\nabla_{\mathbf{x}} \cdot \boldsymbol{\psi}^h, p^h + p')_{\Omega} + (\nabla_{\mathbf{x}} \boldsymbol{\psi}^h, \rho^h \mathbf{b})_{\Omega} + \dots,$$

$$0 = (\phi^h, \dot{p}^h - \tilde{\kappa} \nabla_{\mathbf{x}} \cdot \mathbf{v}^h)_{\Omega} + (\nabla_{\mathbf{x}} \phi^h, \tilde{\kappa} \mathbf{v}')_{\Omega} +$$

$$(\nabla_{\mathbf{x}} \phi^h, \tilde{\kappa}' \nabla_{\mathbf{x}} \cdot \mathbf{v}^h \mathbf{u}') - (\nabla_{\mathbf{x}} \phi^h, \tilde{\kappa} \nabla_{\mathbf{x}} \mathbf{v}^h \mathbf{u}').$$

for all $\boldsymbol{\psi}^h \in \mathcal{V}_v^h$ and $\phi^h \in \mathcal{V}_s^h$.

- $\tilde{\kappa}' = U'''(J)J^2 + U''(J)J = \kappa(J - J^{-1})/2$.
- Fine scale variables ($\tau = \Delta t/2$):

$$\mathbf{v}' = -\frac{\tau}{\rho^h} (\rho^h \dot{\mathbf{v}}^h - \nabla_{\mathbf{x}} p^h - \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}^{dev} - \rho^h \mathbf{b}),$$

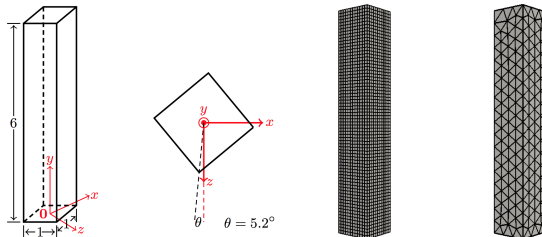
$$\mathbf{u}' = \int_0^t \mathbf{v}'(s) ds, \quad p' = -\tau (\dot{p}^h - \tilde{\kappa} \nabla_{\mathbf{x}} \cdot \mathbf{v}^h).$$



Numerical Examples: Bending Test

Bending Beam

- Problem setup and two meshes.

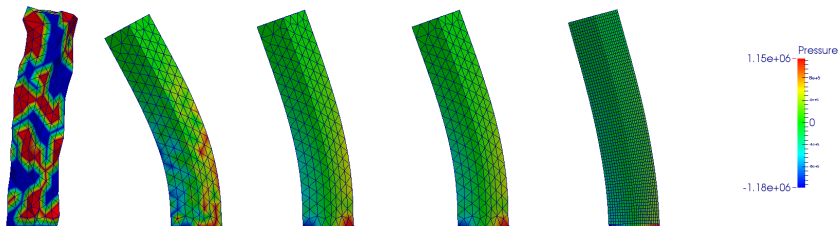


- Tests: $\nu = 0.499$.
- Methods:
 - Standard Galerkin: Q_1 . (Reference.)
 - Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_0$. (Locking!)
 - Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_1$. (Instability.)
 - VMS-FEM: $\mathcal{P}_1/\mathcal{P}_1$ (both versions).

Numerical Examples: Bending Test

Bending Beam

$$\nu = 0.499, t = 1.0s$$



Gal. $\mathcal{P}^1/\mathcal{P}^0$ Gal. $\mathcal{P}^1/\mathcal{P}^1$ VMS¹ $\mathcal{P}^1/\mathcal{P}^1$ VMS² $\mathcal{P}^1/\mathcal{P}^1$ Gal. $\mathcal{Q}^1/\mathcal{Q}^1$

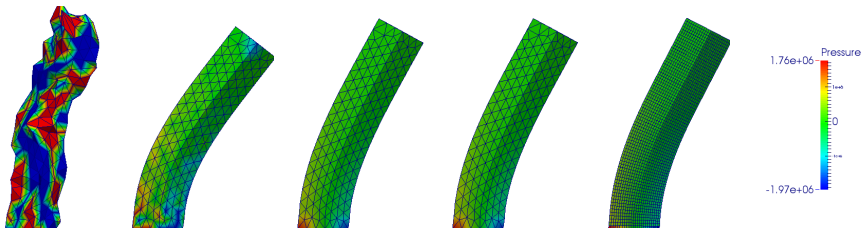
- Tetrahedral mesh: 1,775 elements.
- Hexahedral mesh: $12 \times 12 \times 72$ elements.



Numerical Examples: Bending Test

Bending Beam

$$\nu = 0.499, t = 2.0s$$



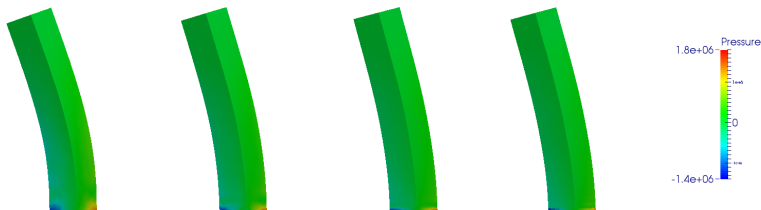
Gal. $\mathcal{P}^1/\mathcal{P}^0$ Gal. $\mathcal{P}^1/\mathcal{P}^1$ VMS¹ $\mathcal{P}^1/\mathcal{P}^1$ VMS² $\mathcal{P}^1/\mathcal{P}^1$ Gal. $\mathcal{Q}^1/\mathcal{Q}^1$

- Tetrahedral mesh: 1,775 elements.
- Hexahedral mesh: $12 \times 12 \times 72$ elements.

Numerical Examples: Bending Test

Bending Beam

$$\nu = 0.499, t = 1.0s$$



VMS¹, mesh 1. VMS¹, mesh 2. VMS¹, mesh 3. Reference.

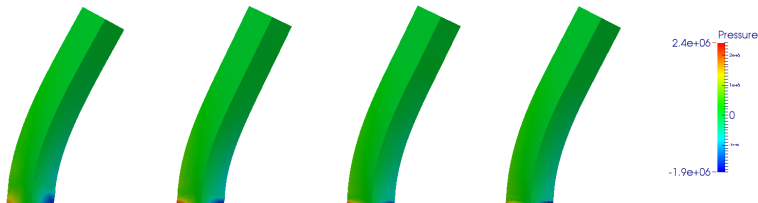
- Tetrahedral meshes: 1,775, 11,575, and 84,042 elements.
- Hexahedral mesh: $12 \times 12 \times 72$ elements.



Numerical Examples: Bending Test

Bending Beam

$$\nu = 0.499, t = 2.0s$$



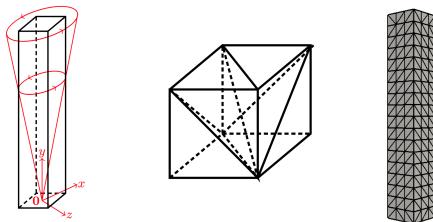
VMS¹, mesh 1. VMS¹, mesh 2. VMS¹, mesh 3. Reference.

- Tetrahedral meshes: 1,775, 11,575, and 84,042 elements.
- Hexahedral mesh: $12 \times 12 \times 72$ elements.

Numerical Examples: Twisting Test

Twisting Beam

- Problem setup and a structured tetrahedral mesh.



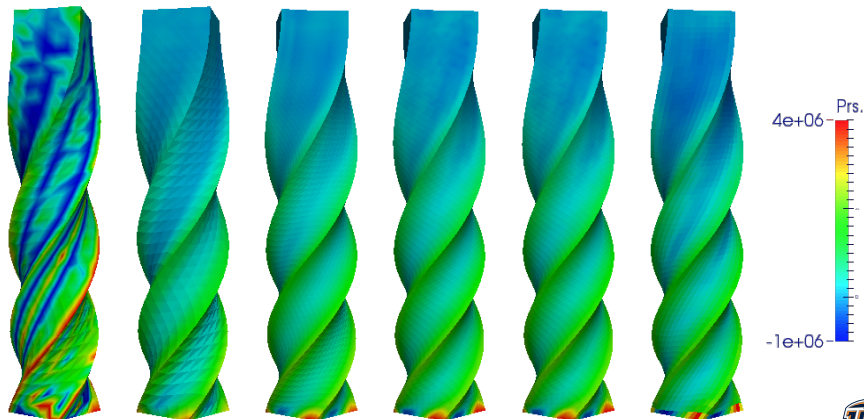
- $\nu = 0.499$.
- Methods:
 - Standard Galerkin: Q_1 , hexahedral mesh. (Reference.)
 - Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_0$, structured tetrahedral mesh. (Fail!)
 - Standard Galerkin: $\mathcal{P}_1/\mathcal{P}_1$, structured tetrahedral mesh. (Instability.)
 - VMS-FEM: $\mathcal{P}_1/\mathcal{P}_1$, structured and unstructured tetrahedral mesh.



Numerical Examples: Twisting Test

Twisting Beam

Pressure at $t = 0.1s$



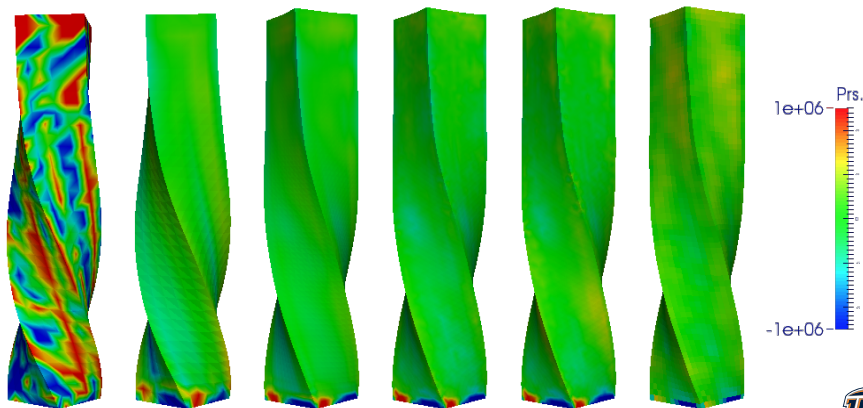
Gal. M1. VMS¹, M1.VMS¹, M2.VMS¹, uns.VMS², uns. Ref.



Numerical Examples: Twisting Test

Twisting Beam

Pressure at $t = 0.2s$



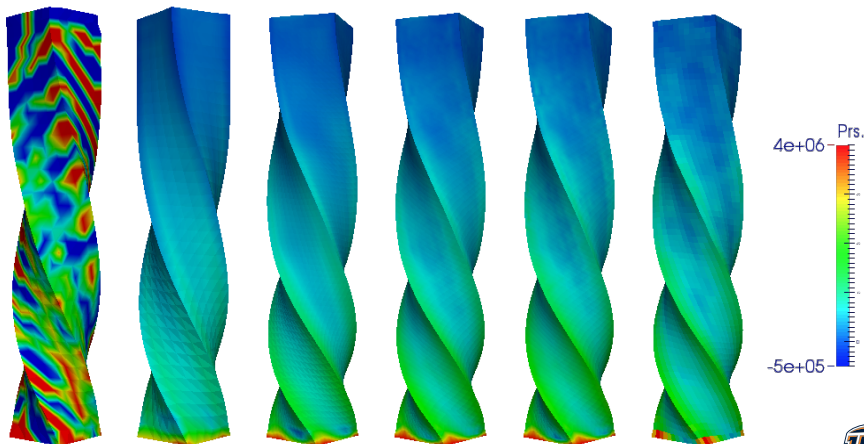
Gal. M1. VMS¹, M1.VMS¹, M2.VMS¹, uns.VMS², uns. Ref.



Numerical Examples: Twisting Test

Twisting Beam

Pressure at $t = 0.3s$



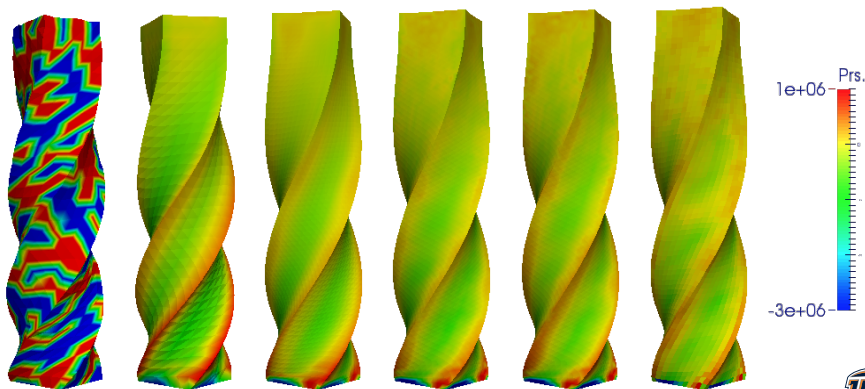
Gal. M1. VMS^1 , M1. VMS^1 , M2. VMS^1 , uns. VMS^2 , uns. Ref.



Numerical Examples: Twisting Test

Twisting Beam

Pressure at $t = 0.4s$



Gal. M1. VMS^1 , $M1.VMS^1$, $M2.VMS^1$, $uns.VMS^2$, $uns.$ Ref.



Outline

- ① Introduction and Motivation
- ② The Variational Multiscale Approach: Overview
- ③ The Variational Multiscale Approach for Solid Mechanics
- ④ Further Applications and Extensions of the VMS-FEM
 - Extension to Other Solids
 - VMS-FEM in Fluid-Structure Interactions



Extension to Viscoelastic Solids

Linear viscoelastic materials: the generalized Maxwell model

- History-dependent stress-strain relation.
- Linear viscoelasticity (only in the deviatoric stress component):

$$\boldsymbol{\sigma}(t) = p(t)\mathbf{I} + \left(g * \frac{d}{dt}(2\mu\tilde{\nabla}^s \mathbf{u}) \right) (t).$$

- Prony series of the relaxation function $g(t)$:

$$g(t) \approx \gamma_\infty + \sum_{i=1}^m \gamma_i e^{-t/\tau_i}; \quad \gamma_\infty, \gamma_i \geq 0, \gamma_\infty + \sum_i \gamma_i = 1, \tau_i > 0.$$

- Introduce m internal state variables:

$$\mathbf{h}^{[i]}(t) \stackrel{\text{def}}{=} \int_{-\infty}^t e^{-(t-s)/\tau_i} \frac{d}{ds} \boldsymbol{\sigma}^{\text{dev},e}(s) ds,$$

$$\Rightarrow \boldsymbol{\sigma}(t) = p(t)\mathbf{I} + \gamma_\infty \boldsymbol{\sigma}^{\text{dev},e}(t) + \sum_{i=1}^m \gamma_i \mathbf{h}^{[i]}(t).$$

- Hyperbolic; the same maximum characteristic speed (bulk wave).



Extension to Viscoelastic Solids

Linear viscoelastic materials: the generalized Maxwell model

- Pressure equation remains the same.
- VMS-FEM: compute the deviatoric stress $\boldsymbol{\sigma}^{dev}$ correctly.
- Inherent dissipation.

$$\begin{aligned}\mathcal{E}(t) &= \left\| \rho_0^{\frac{1}{2}} \mathbf{v}^h \right\|^2 + \left\| \kappa^{-\frac{1}{2}} p^h \right\|^2 + \left\| (\gamma_\infty 2\mu)^{\frac{1}{2}} \tilde{\nabla}^s \mathbf{u}^h \right\|_F^2 + \\ &\quad \sum_{i=1}^m \left\| \left(\frac{\gamma_i}{2\mu} \right)^{\frac{1}{2}} \mathbf{h}^{[i]} \right\|_F^2, \\ \dot{\mathcal{E}} &= - \sum_{i=1}^m \left\| \left(\frac{\gamma_i}{\mu \tau_i} \right)^{\frac{1}{2}} \mathbf{h}^{[i]} \right\|_F^2 \leq 0.\end{aligned}$$

- The effective compressibility changes with time.



Extension to Viscoelastic Solids

Linear viscoelastic materials: Swinging plate

- Method of the manufactured solutions.
- Linear viscoelasticity problem with analytic solution.

$$\mathbf{u}^*(\mathbf{x}, t) = U_0 \sin(\omega t) \begin{bmatrix} -\sin(\theta x) \cos(\theta y) \\ \cos(\theta x) \sin(\theta y) \end{bmatrix}, \quad (x, y) \in [0, 2]^2.$$

- $\mathbf{v}^* = \dot{\mathbf{u}}^*, p^* = 0.$
- Corresponding internal variables:

$$\mathbf{h}^{[i]} = 2\mu\theta U_0 \alpha_i \begin{bmatrix} -\cos(\theta x) \cos(\theta y) & 0 \\ 0 & \cos(\theta x) \cos(\theta y) \end{bmatrix},$$

$$\alpha_i = \frac{\omega \tau_i (\cos(\omega t) + \omega \tau_i \sin(\omega t) - e^{-t/\tau_i})}{1 + \omega^2 (\tau_i)^2}.$$

- Body force:

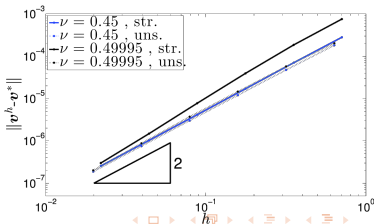
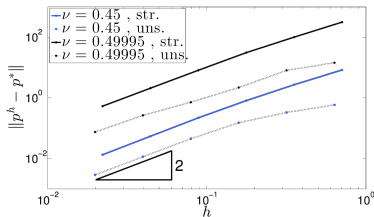
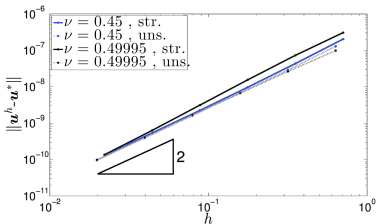
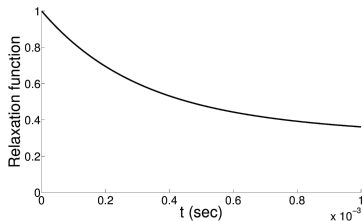
$$\mathbf{b}^* = \left[-\omega^2 + \frac{2\mu\theta^2}{\rho_0} (\gamma_\infty + \sum \gamma_i \frac{\alpha_i}{\sin(\omega t)}) \right] \mathbf{u}^*.$$



Extension to Viscoelastic Solids

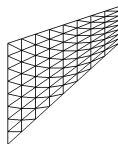
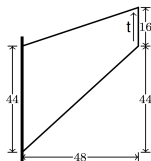
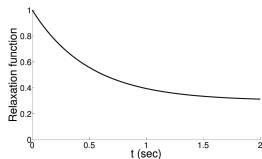
Linear viscoelastic materials: Swinging plate

$$E = 1.7 \times 10^7 \text{ Pa}, \rho_0 = 1.1 \times 10^3 \text{ kg/m}^3, U_0 = 10^{-2} \text{ m}, \theta = \pi/2, \omega = \theta \sqrt{2\mu/\rho_0}.$$

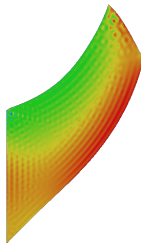
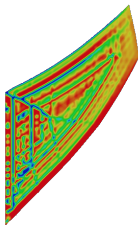


Extension to Viscoelastic Solids

Linear viscoelastic materials: Cook's membrane



- $\mathcal{P}_1/\mathcal{P}_0$ – volume locking; $\mathcal{P}_1/\mathcal{P}_1$ – pressure instability.
- VMS-stabilized $\mathcal{P}_1/\mathcal{P}_1$ – stable solutions.



Extension to Viscoelastic Solids

Linear viscoelastic materials: Cook's membrane



Extension to Viscoelastic Solids

Linear viscoelastic materials: Cook's membrane



Extension to Viscoelastic Solids

Nonlinear viscoelastic materials

- Elasticity:

$$\mathbf{S} = U'(J)J\mathbf{C}^{-1} + J^{-2/3} \text{DEV}_{\mathbf{C}} [2\partial_{\overline{\mathbf{C}}}W(\overline{\mathbf{C}})] .$$

- Viscoelasticity:

$$\mathbf{S}(t) = U'(J)J\mathbf{C}^{-1} + J^{-2/3} \int_{-\infty}^t g(t-s) \frac{d}{ds} (\text{DEV}_{\mathbf{C}(s)} [2\partial_{\overline{\mathbf{C}}}W(\overline{\mathbf{C}}(s))]) ds .$$

- Prony series of $g(t)$, and internal state variables:

$$g(t) = \gamma_{\infty} + \sum_{i=1}^m \gamma_i e^{-t/\tau_i} ,$$

$$\mathbf{H}^{[i]}(t) = \int_{-\infty}^t e^{-(t-s)/\tau_i} \frac{d}{ds} (\text{DEV}_{\mathbf{C}(s)} [2\partial_{\overline{\mathbf{C}}}W(\overline{\mathbf{C}}(s))]) ds ,$$

$$\boldsymbol{\sigma}(t) = U'(J)\mathbf{I} + \gamma_{\infty}J^{-5/3} \text{dev} [2\mathbf{F}\partial_{\overline{\mathbf{C}}}W(\overline{\mathbf{C}})\mathbf{F}^t] +$$

$$J^{-5/3} \sum_{i=1}^m \gamma_i \text{dev} [\mathbf{F}\mathbf{H}^{[i]}\mathbf{F}^t] .$$



Extension to Viscoelastic Solids

Nonlinear viscoelastic materials: Bending beam



Extension to Viscoelastic Solids

Nonlinear viscoelastic materials: Twisting beam



Extension to Elastoplastic Solids

Isotropic elastoplastic materials

- Deviatoric plastic flow.
- Multiplicative elastoplasticity ($\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$).
- The associative von-Mises model:

$$\Phi = \sqrt{\frac{3}{2} \boldsymbol{\tau}^{dev} : \boldsymbol{\tau}^{dev}} - (\sigma_y + Hq).$$



Extension to Elastoplastic Solids

Isotropic elastoplastic materials

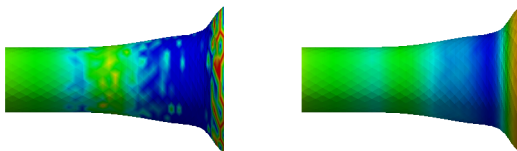
- Deviatoric plastic flow.
- Multiplicative elastoplasticity ($\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$).
- The associative von-Mises model:

$$\Phi = \sqrt{\frac{3}{2} \boldsymbol{\tau}^{dev} : \boldsymbol{\tau}^{dev}} - (\sigma_y + Hq).$$

Taylor bar

- $\rho = 8,930 \text{kg/m}^3$, $E = 117 \text{GPa}$, $\nu = 0.35$, $\sigma_y = 0.4 \text{GPa}$, $H = 0.1 \text{GPa}$.

$v_0 = 227 \text{m/s}$, **pressure at $t = 80 \mu\text{s}$**



Left: Galerkin $\mathcal{P}_1/\mathcal{P}_1$; right: VMS-FEM $\mathcal{P}_1/\mathcal{P}_1$.

Extension to Elastoplastic Solids

Isotropic elastoplastic materials

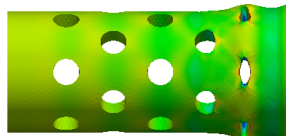
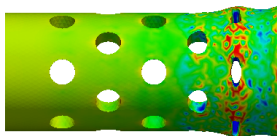
- Deviatoric plastic flow.
- Multiplicative elastoplasticity ($\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$).
- The associative von-Mises model:

$$\Phi = \sqrt{\frac{3}{2} \boldsymbol{\tau}^{dev} : \boldsymbol{\tau}^{dev}} - (\sigma_y + Hq).$$

Modified Taylor “bar”

- $\rho = 8,930 \text{ kg/m}^3$, $E = 117 \text{ GPa}$, $\nu = 0.35$, $\sigma_y = 0.4 \text{ GPa}$, $H = 0.1 \text{ GPa}$.

$v_0 = 100 \text{ m/s}$, **pressure at $t = 30 \mu\text{s}$**



Left: Galerkin $\mathcal{P}_1/\mathcal{P}_1$; right: VMS-FEM $\mathcal{P}_1/\mathcal{P}_1$.

A VMS-FEM for Inviscid Compressible Flows

The Shock Hydrodynamics

- Arbitrary Lagrangian-Eulerian framework.
- Lagrangian plus remap.
- Lagrangian step: VMS-FEM, 2nd-order accuracy, provable energy stability².
- Remap: frame-invariant, 2nd-order accuracy, conservative, monotone³.

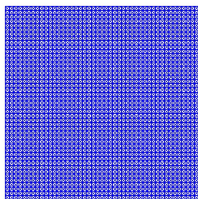


A VMS-FEM for Inviscid Compressible Flows

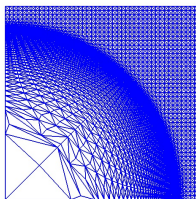
The Shock Hydrodynamics

- Arbitrary Lagrangian-Eulerian framework.
- Lagrangian plus remap.

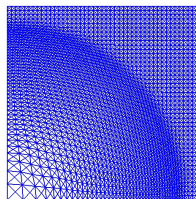
Eulerian



Lagrangian



Lagrangian + Remap



- Lagrangian step: VMS-FEM, 2nd-order accuracy, provable energy stability².
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A VMS-FEM for Inviscid Compressible Flows

The Shock Hydrodynamics

- Arbitrary Lagrangian-Eulerian framework.
- Lagrangian plus remap.
- Lagrangian step: VMS-FEM, 2nd-order accuracy, provable energy stability².

$$\dot{\mathbf{x}} = \mathbf{v},$$

$$\rho J = J_0,$$

$$\rho \dot{\mathbf{v}} = -\nabla_{\mathbf{x}} p + \rho \mathbf{b},$$

$$\dot{p} = -\rho c_s^2 \nabla_{\mathbf{x}} \cdot \mathbf{v}$$

- VMS-approach: $\mathbf{v}'$ and p' .
- Artificial viscosity.
- Artificial heat flux.
- Entropy inequality satisfied.

- Remap: frame-invariant, 2nd-order accuracy, conservative, monotone³.

¹G. Scovazzi. *J. Comp. Phys.*, 2012.

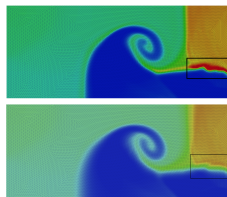
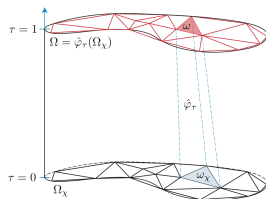
²X. Zeng, G. Scovazzi. *J. Comp. Phys.*, 2014.



A VMS-FEM for Inviscid Compressible Flows

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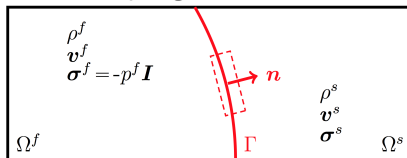


¹G. Scovazzi. *J. Comp. Phys.*, 2012.

²X. Zeng, G. Scovazzi. *J. Comp. Phys.*, 2014.

Coupling the Fluid and the Solid

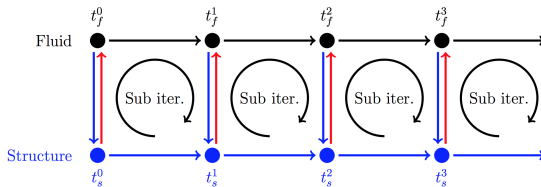
The Two-Way Partitioned Coupling



$$(v^f - v^s) \cdot n = 0$$

$$(\sigma^f - \sigma^s)n = 0 \Rightarrow -p^f n = \sigma^s n$$

- Fluid send pressure to solid.
- Solid send velocity and displacement to fluid.
- Sub-iterations to improve accuracy.



Fluid-Structure Interaction: Examples

Shock-Gel Interaction

- Fluid: a point-source blast wave.

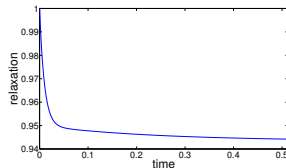
$$\rho = 1.29 \text{kg/m}^3, \gamma = 1.4.$$

- Solid: nonlinear viscoelastic gel.

$$\rho_0 = 1.06 \times 10^3 \text{kg/m}^3,$$

$$E = 65.7 \text{kPa}, \nu = 0.495.$$

Relaxation function $g(t)$.



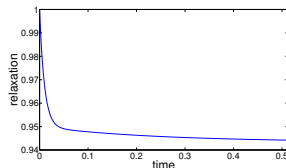
Fluid-Structure Interaction: Examples

Shock-Shell-Gel Interaction

- Fluid: a point-source blast wave.
- Inner solid: nonlinear viscoelastic gel.
- Outer solid: nonlinear elastic shell.

$$\rho_0 = 1.2 \times 10^3 \text{kg/m}^3,$$
$$E = 1.56 \text{GPa}, \nu = 0.37.$$

Relaxation function $g(t)$.



Fluid-Structure Interaction: Examples

In-Tank Blast

- Fluid: air, point-source blast wave.
- Tank: copper, complex geometry.

Pressure evolution until $t = 3.8 \times 10^{-5}\text{s}$

