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Generalizations and New Proof of the Discrete-Time Positive Real Lemma and Bounded Real Lemma

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***Abstract*—There are three different restatements claimed to be equivalent to the definition of discrete-time positive realness (DTPR) in the literature. These restatements were obtained by assuming that they are similar to the results of continuous-time positive realness when the transfer function has poles on the stability boundary. In this paper it is shown that only one of them is equivalent to the DTPR lemma and others are disproved by counter-examples. Furthermore, the DTPR lemma is specialized for minimal systems which have all poles on the unit cycle, the DTPR lemma is also generalized for nonminimal systems, the discrete-time bounded real (DTBR) lemma is proven by a simple method, and then the DTBR lemma is extended to the nonminimal case. Continuous-time results are also briefly considered in the Appendix.**

***Index Terms*— Bounded real lemma, bounded realness, nonminimal systems, positive real lemma, positive realness.**

I. Introduction

The classical (continuous-time) positive real lemma and bounded real lemma are very useful in optimal control, stability analysis, and network synthesis for continuous-time systems [1]. The discrete-time positive real (DTPR) lemma and discrete-time bounded real (DTBR) lemma have also been established in [2]–[5]. The DTPR lemma and DTBR lemma have found applications in stability analysis [2], absolute stability study [5], low-sensitivity filter design [3], limit-cycle free filter synthesis [3], solution to the two-dimensional (2-D) Lyapunov equation [6], and signal processing [7], [8]. However, there are three different restatements which were claimed to provide the necessary and sufficient conditions for a system to be DTPR, and each one of them is distinct from the others when the system has poles on the unit circle. In this paper it is pointed out that two of the three necessary and sufficient conditions are not equivalent to the DTPR lemma and it is shown that there exist generalized versions for the DTPR lemma and the DTBR lemma when the realization of a discrete-time system is nonminimal.

II. Preliminaries

In this Section, we briefly review the definitions of DTPR and bounded realness and some related results in the literature.

Consider a $p \times p$ transfer matrix $H(z)$ of a p-input p-output system having entries which are rational functions in the complex variable z and $H(z)$ has the following form:

$$H(z) = D + C(zI - A)^{-1}B \tag{1}$$

where $A, B, C,$ and D are real constant matrices with appropriate dimensions. Then the definition of DTPR is given by Hitz and Anderson [2] as follows.

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Definition 1 [2]: Let $H(z)$ be a square matrix of real rational functions. Then $H(z)$ is called DTPR if it has the following properties.

i) All the elements of $H(z)$ are analytic in $|z| > 1$.
ii) $H^t(z^*) + H(z) \geq 0$ for all $|z| > 1$.

Applying bilinear transformation to the CTPR lemma [1], Hitz and Anderson proved the following lemma.

Lemma 1 [2]: Let $H(z)$ be a square matrix of real rational function of z with no poles in $|z| > 1$ and simple poles only on $|z| = 1$ and let (A, B, C, D) be a minimal realization of $H(z)$. Then necessary and sufficient conditions for $H(z)$ to be DTPR are that there exist a real symmetric positive definite matrix P and real matrices L and W such that

$$P - A^tPA = L^tL \tag{2a}$$

$$C^t - A^tPB = L^tW \tag{2b}$$

$$D^t + D - B^tPB = W^tW. \tag{2c}$$

Regarding Definition 1, three restatements were proposed in the literature as follows.

Lemma 2 [2]: A square matrix $H(z)$ whose elements are real rational functions analytic in $|z| > 1$ is DTPR if, and only if, it satisfies all the following conditions.

i) All poles of each entry of $H(z)$ on $|z| = 1$ are simple.
ii) $H^t(e^{-j\theta}) + H(e^{j\theta}) \geq 0$ for all real θ at which $H(e^{j\theta})$ exists.
iii) If $z_0 = e^{j\theta_0}$, θ_0 real, is a pole of an entry of $H(z)$, and if K_0 is the residue matrix of $H(z)$ at $z = z_0$, then the matrix $e^{-j\theta_0}K_0$ is nonnegative definite Hermitian.

Lemma 3 [4], [5]: A square matrix $H(z)$ of real rational functions is a DTPR if and only if the following pertains.

i) $H(z)$ has elements analytic in $|z| > 1$.
ii) The poles of the elements of $H(z)$ on $|z| = 1$ are simple and the associated residue matrices of $H(z)$ at these poles are positive semidefinite.
iii) $H(e^{j\theta}) + H^t(e^{-j\theta}) \geq 0$ for all real θ for which $H(e^{j\theta})$ exists.

Lemma 4 [8, pp. 245, 246]: $H(z)$ is positive real if and only if the following conditions hold.

i) $H(e^{j\theta}) + H^t(e^{-j\theta}) \geq 0$, $\theta \in [0, 2\pi]$, such that $e^{j\theta}$ is not a pole of $H(z)$.
ii) No poles of $H(z)$ lie in $|z| > 1$.
iii) If $z_0 = e^{j\theta_0}$ is a pole of $H(z)$ such that $|z_0| = 1$, then it is not a repeated pole and $\lim_{z\to z_0}(z - z_0)H(z) = M/(z_0 - \overline{z}_0)$ where M is real and positive (definite).

It is important to note that Lemmas 2–4 are not agreeable with one another when $H(z)$ has poles on the unit circle. We show here by examples that Lemmas 3 and 4 are not equivalent to Definition 1, Lemmas 1 and 2. Consider three functions as follows:

$$H_1(z) = 1 - \frac{1.5}{z+1}, \qquad H_2(z) = 1 + \frac{1}{z+1},$$
$$H_3(z) = 1 - \frac{0.5z + 1}{z^2 + z + 1}.$$

Obviously, all the above three functions are analytic in $|z| > 1$. However, after testing the positive realness of $H_1(z)$, $H_2(z)$, and

TABLE I
COMPARISON RESULTS

	$H_1(z)$	$H_2(z)$	$H_3(z)$
Definition 1	PR	not PR	PR
Lemma 1	PR	not PR	PR
Lemma 2	PR	not PR	PR
Lemma 3	not PR	PR	not PR
Lemma 4	not PR	not PR	not PR

$H_3(z)$ by Definition 1, Lemmas 1–4 we have different results, as summarized in Table I.

As can be seen from Table I, Definition 1, Lemmas 1 and 2 are agreeable with each other. However, neither Lemmas 3 nor 4 are agreeable with Definition 1, Lemmas 1 and 2, where Lemma 1 plays a central role in various areas of research, as was mentioned before.

Remark 1: It is also worthwhile to emphasize that if matrix $H(z)$ cannot be expressed as (1) then Definition 1 is not equivalent to Lemma 1 either. For example, $H(z) = 1 - z$, which is a rational function but cannot be expressed as (1), we have $H(\beta e^{-j\theta}) + H(\beta e^{j\theta}) = 2(1 - \beta\cos\theta)$ and $H(e^{-j\theta}) + H(e^{j\theta}) = 2(1 - \cos\theta)$. Clearly, $H(z)$ is not DTPR in the sense of Definition 1 and $H(z)$ is DTPR in the sense of Lemma 1.

Before we finish this Section, we quote the definition of DTBR as follows [1], [3].

Definition 2: Let $H(z)$ be a $p \times q$ $(p \geq q)$ transfer matrix. Then $H(z)$ is called DTBR if the following pertains.

i) All poles of each entry of $H(z)$ lie in $|z| < 1$.
ii) $I - H^t(z^{-1})H(z) \geq 0$ for all $|z| = 1$.

III. THE DISCRETE-TIME POSITIVE REAL LEMMA

In this Section we develop the DTPR lemma for general (including minimal and nonminimal) realization systems. Lemma 5 is for minimal system $H(z)$ which has all poles on $|z| = 1$ and Lemma 6 is for general system $H(z)$ without reachability and observability constraints. We call this lemma as generalized discrete-time positive real (GDTPR) lemma.

Lemma 5: Let the square transfer matrix $H(z)$ have all poles on $|z| = 1$ and let (A, B, C, D) be a minimal realization of $H(z)$. Then $H(z)$ is positive real if and only if there exists real matrix W and real symmetric matrix $P > 0$ such that

$$P - A^tPA = 0 \tag{3a}$$
$$C^t - A^tPB = 0 \tag{3b}$$
$$D^t + D - B^tPB = W^tW. \tag{3c}$$

Proof: Since $H(z)$ has all poles on $|z| = 1$ and (A, B, C, D) is a minimal realization of $H(z)$, if all poles of $H(z)$ are simple then, without loss of generality, there always exists a real nonsingular matrix T such that

$$\hat{A} = T^{-1}AT = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & \ddots & \\ & & & & A_m \end{bmatrix},$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \cdots \\ B_m \end{bmatrix} \tag{4a}$$

$$\hat{C} = CT = [C_1 \quad C_2 \quad C_3 \quad \cdots \quad C_m] \tag{4b}$$

where

$$A_1 = -1, \qquad A_2 = 1, \qquad A_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \quad i = 3, \cdots, m \tag{5}$$

with $\cos\theta_i \neq \pm 1$ for $i = 3, \cdots, m$; $\cos\theta_i \neq \cos\theta_j$ for $i \neq j$. Since $H(z)$ is a minimal system, so $C_kB_k \neq 0$, $C_kA_k^tB_k \neq 0$, $k = 1, \cdots, m$, otherwise the related A_k does not exist. From (4) $H(z)$ can be expressed as

$$H(z) = D + C_1(z+1)^{-1}B_1 + C_2(z-1)^{-1}B_2 + \sum_{i=3}^{m} C_i(zI - A_i)^{-1}B_i. \tag{6}$$

If we assume that K_1, K_2, and K_i, $i = 3, \cdots, m$ are the residue matrices of $H(z)$ at $z = -1$, $z = 1$ and $z = e^{j\theta_i}$, $i = 3, \cdots, m$, respectively, then we have

$$K_1 = C_1B_1, \qquad K_2 = C_2B_2, \qquad K_i = \tfrac{1}{2}C_i\begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}B_i \tag{7}$$

and

$$\begin{aligned} e^{-j\theta_i}K_i &= \tfrac{1}{2}C_i\begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}B_i + \frac{j}{2}C_i\begin{bmatrix} -\sin\theta_i & -\cos\theta_i \\ \cos\theta_i & -\sin\theta_i \end{bmatrix}B_i \\ &= \frac{1}{2}C_iA_i^tB_i + \frac{j}{2}C_i\begin{bmatrix} -\sin\theta_i & -\cos\theta_i \\ \cos\theta_i & -\sin\theta_i \end{bmatrix}B_i. \end{aligned} \tag{8}$$

Necessity: Suppose that $H(z)$ is DTPR. We proceed to (3). Based on iii) of Lemma 2, we have three cases as follows.

i) For $A_1 = -1$ and $C_1B_1 \neq 0$ the necessary and sufficient condition for $-K_1 = -C_1B_1 \geq 0$ is $C_1 = -B_1^t\rho_1$ with $\rho_1 > 0$ and therefore
$$A_1^tP_1B_1 = C_1^t, \qquad P_1 - A_1^tP_1A_1 = 0 \tag{9}$$
with $P_1 = \rho_1$.

ii) For $A_2 = 1$, similarly to i), $K_2 = C_2B_2 \geq 0$ if and only if $C_2 = B_2^t\rho_2$ with $\rho_2 > 0$ and so
$$A_2^tP_2B_2 = C_2^t, \qquad P_2 - A_2^tP_2A_2 = 0 \tag{10}$$
with $P_2 = \rho_2$.

iii) For
$$A_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$$
with $\cos\theta_i \neq \pm 1$, noting (8), the necessary and sufficient condition for $e^{-j\theta_i}K_i$ being nonnegative definite Hermitian is
$$C_iA_i^tB_i \geq 0 \tag{11a}$$
$$e^{-j\theta_i}K_i + e^{j\theta_i}K_i^H \geq 0. \tag{11b}$$

The equivalent condition for (11a) $C_iB_i \neq 0$ and $C_iA_i^tB_i \neq 0$ is $C_i^t = P_iA_i^tB_i$ with $P_i = P_i^t > 0$ for both (11a) and (11b). Then $P_i = \rho_iI_2$ with $\rho_i > 0$. Thus, we have
$$A_i^tP_iB_i = C_i^t, \qquad P_i - A_i^tP_iA_i = 0 \tag{12}$$
with $P_i = \rho_iI_2$, $i = 3, \cdots, m$.

According to (5), (6), (9), (10), and (12) we have

$$\begin{aligned} &H^t(z^{-1}) + H(z) \\ &\quad = D^t + D - B_1^t P_1 B_1 - B_2^t P_2 B_2 \\ &\qquad + \sum_{i=3}^{m} B_i^t \, \frac{\rho_i(zA_i - I_2 + zA_i^t - z^2 I_2)}{z^2 - 2z\cos\theta_i + 1} \, B_i \\ &\quad = D^t + D - B_1^t P_1 B_1 - B_2^t P_2 B_2 - \sum_{i=3}^{m} B_i^t P_i B_i \\ &\quad = D^t + D - \hat{B}\hat{P}\hat{B} \end{aligned} \tag{13}$$

where $\hat{P} = P_1 \oplus P_2 \oplus \cdots \oplus P_m$. Based on ii) of Lemma 2 and (13), there always exists a real matrix W such that

$$D^t + D - \hat{B}\hat{P}\hat{B} = W^t W. \tag{14}$$

From (9), (10), and (12) we have

$$\hat{A}^t \hat{P} \hat{A} - \hat{P} = 0, \qquad \hat{A}^t \hat{P} \hat{B} - \hat{C}^t = 0. \tag{15}$$

Based on (4), we can obtain (3) from (14) and (15) with $P = T^{-t}\hat{P}T^{-1}$. This completes the necessity proof.

Sufficiency: Based on (3a), $P > 0$ and $H(z)$ being a minimal system, we can conclude that all poles of $H(z)$ are on $|z| = 1$ and simple. Then, we can prove the sufficiency part of this lemma by the following equation:

$$\begin{aligned} &(z^* I - A^t)P(zI - A) + (z^* I - A^t)PA + A^t P(zI - A) \\ &\quad = |z|^2 P - A^t P A. \end{aligned} \tag{16}$$

Premultiplying by $B^t(z^* I - A^t)^{-1}$ and postmultiplying by $(zI - A)^{-1}B$ on (16), together with (3), we have

$$\begin{aligned} &D^t + B^t(z^* I - A^t)^{-1} C^t + D + C(zI - A)^{-1} B \\ &\quad = (|z|^2 - 1) B^t (z^* I - A^t)^{-1} P (zI - A)^{-1} B + W^t W \\ &\quad \geq 0, \qquad \text{for } |z| > 1. \end{aligned} \tag{17}$$

Clearly, $H^t(z^*) + H(z) \geq 0$ in $|z| > 1$. Therefore, $H(z)$ is positive real. □□□

Remark 2: As can be seen from the above proof, (9), (10), and (12) are obtained by using $e^{-j\theta_i} K_i \geq 0$. However, they cannot be obtained by using $K_i \geq 0$ which was stated in Lemma 3. Moreover, they cannot be obtained by the condition iii) of Lemma 4 either.

Remark 3: Based on all of (3), we can directly obtain items ii) and iii) of Lemma 2. Details are omitted here.

Lemma 6—The GDTPR Lemma: Let (A, B, C, D) be a general realization (including minimal and nonminimal cases) of a square transfer function matrix $H(z)$ and let $\mathcal{C}$ be the corresponding reachability matrix given by

$$\mathcal{C} = [B \quad AB \quad \cdots \quad A^{n-1}B] \tag{18}$$

where n is the dimension of square matrix A. Then $H(z)$ is positive real if and only if there exist real matrices L and W and a real symmetric matrix P with $\mathcal{C}^t P \mathcal{C} \geq 0$ such that

$$\mathcal{C}^t (A^t P A - P + L^t L)\mathcal{C} = 0 \tag{19a}$$

$$\mathcal{C}^t (A^t P B + L^t W - C^t) = 0 \tag{19b}$$

$$D^t + D - B^t P B - W^t W = 0. \tag{19c}$$

Proof: This lemma can be proven by using Lemma 5 and the Kalman canonical decomposition [10]: details are omitted here.

Remark 4: In this lemma, if (A, B) is completely reachable then $P \geq 0$ and the reachability matrix $\mathcal{C}$ in (19) can be removed. If (A, B) is not completely reachable, then P could be indefinite. It should be emphasized that the unreachable and/or unobservable states of the realization (A, B, C, D) can be either stable or unstable and this will not affect the above lemma.

IV. The Discrete-Time Bounded Real Lemma

In this Section we first present a simple and elegant method to prove the necessity part of the DTBR lemma and then generalize this lemma for discrete-time nonminimal systems.

Lemma 7—The DTBR Lemma [3]: Let the real matrices (A, B, C, D) be a minimal realization of the $p \times q$ $(p \geq q)$ transfer matrix $H(z)$. Then $H(z)$ is bounded real if and only if there exist real matrices L and W and a real symmetric positive definite matrix P such that

$$A^t P A + C^t C + L^t L = P \tag{20a}$$

$$B^t P B + D^t D + W^t W = I \tag{20b}$$

$$A^t P B + C^t D + L^t W = 0. \tag{20c}$$

Proof: The sufficiency part was proven in [11] with a simple method. We now provide a new proof for the necessity part of this lemma.

Suppose $H(z)$ is bounded real. Based on the spectral factorization [9] there always exists a rational function $K(z) = W + L(zI - A)^{-1}B$ such that [6]

$$I - H^t(z^{-1})H(z) = K^t(z^{-1})K(z) \tag{21}$$

is satisfied.

Equation (21) gives after some algebraic manipulations

$$\begin{aligned} &I - D^t D - W^t W \\ &\quad = (D^t C + W^t L)(zI - A)^{-1} B + B^t (z^{-1} I - A^t)^{-1} \\ &\qquad \cdot (C^t D + L^t W) + B^t (z^{-1} I - A^t)^{-1} \\ &\qquad \cdot (C^t C + L^t L)(zI - A)^{-1} B. \end{aligned} \tag{22}$$

Let $P = \sum_{i=0}^{\infty} A^{t^i}(C^t C + L^t L)A^i$, then P satisfies

$$P - A^t P A = C^t C + L^t L. \tag{23}$$

Comparing the constant terms and the coefficients of z^{i+1} terms in (22), respectively, we get the following equations:

$$I - D^t D - W^t W = B^t P B \tag{24}$$

$$B^t A^{t^i}(A^t P B + C^t D + L^t W) = 0, \qquad i = 0, 1, 2, \cdots. \tag{25}$$

Since (A, B) is completely reachable, (25) gives

$$A^t P B + C^t D + L^t W = 0. \tag{26}$$

Since $H(z)$ is minimal and bounded real, all eigenvalues of A are within the open unit circle and (C, A) is observable. Therefore, P given by (23) is positive definite and, obviously, (24) and (26) are identical to (20b) and (20c), respectively. This completes the proof for the necessity part of this lemma. □□□

It is worthwhile to note that our above proof method is a derivation-based method rather than a verification-based method. This means that it can be used to derive new necessary conditions, rather than only verifying existing conditions. For example, we successfully applied this method to get necessary and sufficient conditions for discrete-time lossless bounded real lemmas of multidimensional digital systems [12] and to derive some properties of wide positive realness and wide strict positive realness [13].

Remark 5: If $H(z)$ is lossless bounded real, i.e., $I - H^t(z^{-1})H(z) = 0$, then $K(z) = 0$ with $L = 0$ and $W = 0$. In this case, this lemma becomes the DTLBR lemma: details are omitted for brevity.

We are now in a position to establish the generalized discrete-time bounded real (GDTBR) lemma as follows.

Lemma 8—The GDTBR Lemma: Let the real matrices (A, B, C, D) be a general realization (including minimal and nonminimal cases) of the $p \times q$ $(p \geq q)$ transfer matrix $H(z)$ and let $\mathcal{C}$ be the related reachability matrix defined by (18). Then $H(z)$ is bounded real if and only if there exist real matrices L and W and a real symmetric matrix P with $\mathcal{C}^t P \mathcal{C} \geq 0$ such that

$$\mathcal{C}^t(A^t PA - P + C^t C + L^t L)\mathcal{C} = 0 \tag{27a}$$

$$\mathcal{C}^t(A^t PB + L^t W + C^t D) = 0 \tag{27b}$$

$$D^t D + B^t PB + W^t W = I. \tag{27c}$$

Proof: The proof of this lemma is similar to that of Lemma 6; details are omitted here for brevity.

V. Conclusion

In this paper it is first observed that there are three different restatements which were claimed to be equivalent to the definition of DTPR. It is shown by counter-examples that two of the restatements are incorrect when the system has poles on the unit circle. It can be concluded from Lemma 2 that the conditions regarding the residue matrices at the poles on the unit circle have a different form from their counterpart of the continuous-time case. The DTPR lemma is specialized for minimal systems which have all poles on the unit cycle. Generalized versions of both the DTPR lemma and bounded real lemma are presented for a general realization (including minimal and nonminimal cases) of discrete-time systems. A simple and elegant proof is also given for the necessity part of the DTBR lemma of minimal systems. Continuous-time results for nonminimal systems are briefly considered in the Appendix.

Appendix

In this Appendix, we present the generalized continuous-time positive real (GCTPR) lemma and the generalized continuous-time bounded real (GCTBR) lemma for continuous-time nonminimal systems. Consider a continuous-time transfer function matrix $H(s)$ given by

$$H(s) = D + C(sI - A)^{-1}B \tag{A.1}$$

where A, B, C, and D are real constant matrices. Based on the definitions of positive realness and bounded realness and their related lemmas for minimal systems [1], we can prove the following lemmas.

GCTPR Lemma: Let (A, B, C, D) be a general realization (including minimal and nonminimal cases) of a square transfer matrix $H(s)$ expressed by (A.1) and let $\mathcal{C}$ be the controllability matrix given by

$$\mathcal{C} = [B \quad AB \quad \cdots \quad A^{n-1}B] \tag{A.2}$$

where n is the dimension of square matrix A. Then $H(s)$ is positive real if and only if there exist real matrices L and W and real symmetric matrix P with $\mathcal{C}^t P \mathcal{C} \geq 0$ such that

$$\mathcal{C}^t(A^t P + PA + L^t L)\mathcal{C} = 0 \tag{A.3a}$$

$$\mathcal{C}^t(PB - C^t + L^t W) = 0 \tag{A.3b}$$

$$D^t + D - W^t W = 0. \tag{A.3c}$$

GCTBR Lemma: Let (A, B, C, D) be the general realization (including minimal and nonminimal cases) of a square transfer matrix $H(s)$ expressed by (A.1) and let $\mathcal{C}$ be the controllability matrix given by (A.2). Then $H(s)$ is bounded real if and only if there exist real matrices L and W and real symmetric matrix P with $\mathcal{C}^t P \mathcal{C} \geq 0$ such that

$$\mathcal{C}^t(A^t P + PA + C^t C + L^t L)\mathcal{C} = 0 \tag{A.4a}$$

$$\mathcal{C}^t(PB + C^t D + L^t W) = 0 \tag{A.4b}$$

$$I - D^t D - W^t W = 0. \tag{A.4c}$$

It should be pointed out that another GCTPR lemma was presented in [14], and our GCTPR lemma in this appendix is a modified and improved version of [14].

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