Wesley R. Perkins – Research Statement

This statement presents an overview of my research as a whole, a detailed description of my research projects, and a brief description of my current and future research goals, wherein I highlight aspects that would be appropriate for undergraduate- and Master's-level research.

Overview

My research focuses on the existence, stability, and dynamics of nonlinear wave solutions to partial differential equations (PDEs) arising from physical applications. In particular, I am interested in studying nonlinear waves and coherent structures that are motivated by experimental or numerical observations. Several questions arise when a PDE is presented as a model for a physical system:

> Does an idealized version of the observed pattern *exist* as a solution to the PDE? Is such a solution *stable*, i.e. is it able to persist when subjected to small perturbations? What are the *dynamics* of such a solution within the PDE as time evolves? Can the PDE predict *new observations*? Can it explain *why* we see some patterns and not others?

The first three questions endeavor to determine, among other things, whether or not the PDE is a good model for the physical system, and their answers can gauge the strengths and weaknesses of the PDE and inform what (if anything) needs to change in the model. The last two questions seek to understand what the mathematics can teach us about the underlying physical application, and the answers to these questions can motivate new experimental and numerical research. Additionally, the stability of a particular class of solutions is of fundamental practical importance as solutions which are unstable do not naturally arise in applications except possibly as transient phenomena. In particular, understanding why some patterns are stable and others are unstable can motivate methods of stabilizing unstable patterns, which is key in many applications where there exist unstable, yet desirable, structures.

There has historically been significant interest in the stability theory of asymptotically constant structures, such as solitons. More recently, there has been a growing community of mathematicians interested in the stability of spatially-periodic structures. These are often idealized versions of physically-observable patterns which are almost spatially periodic, in the sense that their internal wavelength is much smaller than the size of the physical domain; hence, they may be modeled as exact periodic solutions on unbounded domains to eliminate the influence of far away boundaries. Applications where such patterns exist are numerous and include surface and internal water wave propagation, optical signal propagation, plasma and astrophysics, and inclined thin film flow.

One powerful tool used to study such periodic structures is known as *Whitham's theory of wave modulations*, sometimes referred to as Whitham theory. Whitham theory is a formal, physically-motivated theory used to understand the stability and dynamics of periodic waves when in the presence of perturbations that modulate their fundamental characteristics, such as amplitude or frequency. Whitham theory lacks rigorous justification in general, leading to the open research problem of establishing such rigorous justification. Nevertheless, Whitham theory is commonly used by applied mathematicians and physicists, and its predictions are nearly universally accepted.

To study the stability of periodic structures using rigorous mathematics, as opposed to the formal asymptotic methods used in Whitham theory, one must start by choosing an appropriate function space to encode the class of perturbations being considered. There are two important and widely studied classes of perturbations that arise naturally in applications. If the nonlinear wave or coherent structure is T-periodic, then one may consider *subharmonic* perturbations, i.e. NT-periodic perturbations for some positive integer N, or *localized* perturbations, i.e. perturbations which are integrable on the line. Previous results concerning the class of subharmonic perturbations are non-uniform in N, in the sense that they are degenerate in the limit $N \to \infty$. In particular, the size of the perturbation decreases as $N \to \infty$, which is undesirable in situations where (arbitrarily) large values of N are of physical interest.

It might appear, at first glance, that Whitham theory, subharmonic perturbations, and localized perturbations have nothing to do with each. However, I have shown that the study of localized perturbations is *fundamental* to the rigorous justification of Whitham theory. Moreover, I have developed *new methodologies* that establish a deep connection between localized and subharmonic perturbations. My research has focused on exploring the connections between seemingly disparate mathematical theories. My research results may be broadly summarized as follows:

1. **Modulations of Viscous Fluid Conduit Waves**

   Jointly with Prof. Mathew Johnson, I have rigorously validated the predictions of Whitham theory in the context of the conduit equation [9]. The conduit equation is a nonlinear dispersive PDE governing the evolution of the circular interface separating a light, viscous fluid rising buoyantly through a heavy, more viscous, miscible fluid at small Reynolds numbers [11]. Physical experiments conducted in [12] demonstrate the existence of waves that are locally periodic but whose fundamental characteristics modulate over large space and time scales. Such a wave may be modeled as a perfectly periodic wave subjected to a perturbation that modulates its fundamental characteristics. Whitham theory uses an averaged system to predict whether or not a periodic wave is stable to such perturbations. By validating the predictions of Whitham theory, we provide theoretical support to the recent experimental and numerical results in [12] and [11], respectively.

2. **Subharmonic Dynamics of Periodic Waves in Dissipative Systems**

   Jointly with Prof. Mariana Haragus[1] and Prof. Mathew Johnson, I have studied the linear dynamics of spectrally stable T-periodic stationary wave solutions of the Lugiato-Lefever equation (LLE) [7]. The LLE is a damped, forced NLS-type equation that is widely used to investigate the dynamical properties of laser fields confined in nonlinear optical resonators [3]. It has been shown that such T-periodic solutions are nonlinearly stable to NT-periodic, i.e. subharmonic, perturbations for each $N \in \mathbb{N}$ [15]. Unfortunately, the rate of decay and the allowable size of the initial perturbations both tend to 0 as $N \to \infty$ so that this result is non-uniform in N and is, in fact, empty in the limit $N = \infty$. We introduce a methodology by which a uniform in N stability result may be achieved at the linear level. The obtained uniform decay rates are shown to agree precisely with the decay rates of localized, i.e. integrable on the line, perturbations. In this work, we unify and expand on several existing results concerning the stability and dynamics of such waves, and we set forth a general methodology for studying similar problems, at least at the *linear* level, in other contexts.

   Interestingly, we were unable to push the above analysis to the nonlinear level due to an unavoidable loss of derivatives that occurs in our iteration scheme. If the PDE has dissipation in the highest-order term, one can regain these lost derivatives through a technique known as

[1]FEMTO-ST institute, Univ. Bourgogne-Franche Comté

"nonlinear damping." Unfortunately, the dissipation in the LLE occurs in the lowest-order term, leaving no hope of regaining derivatives through a nonlinear damping estimate. While we continue to research potential solutions to this problem (see the section on my current and future research directions below), Prof. Johnson and I explored if, in the presence of a nonlinear damping estimate, the methodology we developed for the linear analysis could be used to develop an analogous nonlinear stability result which is uniform in N. To that end, we investigated the stability and nonlinear dynamics of spectrally-stable wave trains in reaction-diffusion systems in [10]. Using the nonlinear damping estimate present in reaction-diffusion systems, we were able to successfully introduce a methodology by which a stability result for subharmonic perturbations which is uniform in N may be achieved at the *nonlinear* level.

In the next two sections, I will describe in detail the projects that I outlined above. I will conclude by briefly describing my current and future research goals and highlighting aspects of my research that are amenable to undergraduate- and Master's-level research projects.

1 Modulations of Viscous Fluid Conduit Waves

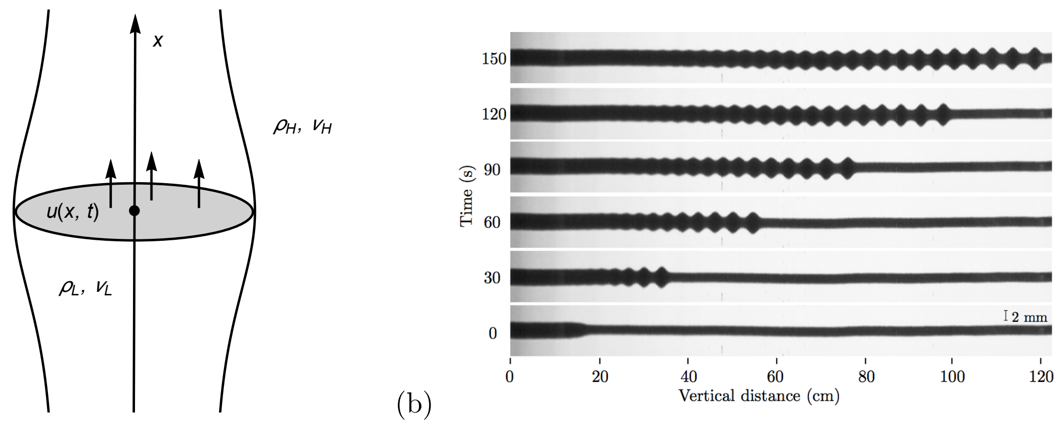
We consider the modulational stability of periodic traveling wave solutions to the conduit equation

$$u_t + (u^2)_x - (u^2(u^{-1}u_t)_x)_x = 0, \tag{1.1}$$

which was derived in [13] to model the evolution of a circular interface separating a light, viscous fluid rising buoyantly through a heavy, more viscous, miscible fluid at small Reynolds numbers [11] (see Figure 1(a)). In (1.1), $u = u(x,t)$ denotes a nondimensional cross-sectional area of the interface at nondimensional vertical coordinate x and nondimensional time t. The conduit equation is also a specific case of the family of magma equations, where it may also be used to model the evolution of magma as it rises buoyantly through a porous, deformable rock matrix [14].

Physical experiments conducted by Mark Hoefer and his team in [12] demonstrate the existence of a coherent structure sometimes referred to as a modulated periodic wave (see Figure 1(b)), which may be described as a wave that is locally periodic but whose fundamental characteristics, such as amplitude and wave number, actually modulate over large space and time scales. There is a well developed theory used to describe the stability of periodic waves to such modulational perturbations. This theory was developed by Whitham in the 1970s and is known as Whitham's theory of wave modulations, or simply Whitham theory. Recently Maiden and Hoefer used Whitham theory to numerically predict when a given periodic wave will be stable/unstable to modulational perturbations [11]. Whitham theory is based on formal asymptotic (WKB) methods, which means its predictions lack rigorous justification in general. This motivates a desire to rigorously justify the predictions of Whitham theory in the context of the conduit equation and thereby justify the results of [11].

The approach of Whitham theory is to express (1.1) in the slow variables $(X,S) := (\varepsilon x, \varepsilon t)$ and carry out a formal WKB expansion as $\varepsilon \to 0$. This yields a closed set of linear homogenized equations, known as the Whitham averaged system, describing the slow evolution of the wave number k and the two conserved quantities M and Q associated with (1.1). The stability of the underlying wave in the Whitham averaged system, considered as a constant in the slow variables, can be determined by linearizing the averaged system about the underlying periodic wave and



(a) (b)

Figure 1: (a) A schematic drawing for the conduit equation. In the physical system, denoting the densities and viscosities of the heavy (outer) and light (inner) fluids as ρ_H, ν_H and ρ_L, ν_L, respectively, the conduit equation holds under the assumption that $\rho_H > \rho_L$ and $\nu_H \gg \nu_L$. The arrows represent rising due to buoyancy. (b) The formation of a modulated periodic wave train propagating in a physical experiment - see [12].

determining its spectrum via the Fourier transform. It is formally expected that a necessary condition for spectral stability of the underlying periodic wave occurs when the spectrum is all real, i.e. when the quasilinear Whitham averaged system is hyperbolic. On the other hand, we expect a sufficient condition for spectral instability occurs when any of the spectrum has a non-trivial imaginary component, i.e. when the quasilinear Whitham averaged system is elliptic. Our goal is to rigorously establish that the spectrum of the quasilinear Whitham averaged system does, in fact, predict whether or not a given periodic wave is stable/unstable to modulational perturbations.

With this motivation in hand, standard phase plane analysis implies that periodic traveling wave solutions of (1.1) of the form $u(x,t) = \phi(x - ct)$ exist for some wave profile $\phi(\cdot) > 0$ and wave speed $c > 0$. By reducing to quadrature, we find that the wave profiles ϕ must satisfy

$$\frac{1}{2}(\phi')^2 = E - \left(\frac{1}{c}\phi^2 \ln|\phi| + a\phi^2 + \phi\right) \tag{1.2}$$

where a and E are constants of integration. In particular, we show that (1.1) admits (modulo translation) a three-parameter family, in fact a C^1 manifold, of periodic traveling wave solutions bounded away from zero. It is important to note that the Whitham averaged system is described in terms of "physical quantities," namely the wave number k and the conserved quantities M and Q, while our existence theory was in terms of "mathematical quantities," namely the wave speed c and the constants of integration a and E. Under the assumption that the Jacobian satisfies

$$\frac{\partial(k, M, Q)}{\partial(a, E, c)} \neq 0,$$

we may translate the rigorous theory into the proper coordinate system, which opens up the possibility of comparing the two theories. This assumption is considered to hold generically, and it is equivalent to the assumption that the quasilinear Whitham system is of an evolutionary type.

In order to rigorously establish that hyperbolicity/ellipticity of the quasilinear Whitham system has the expected connection to spectral stability/instability, we linearize (1.1) about an arbitrary periodic wave profile ϕ, which yields the linear operator $A[\phi]$. We then consider the $L^2(\mathbb{R})$, i.e. the localized, spectrum of $A[\phi]$ near the origin, as this is where information about stability to modulational perturbations is stored. By Floquet theory, this spectrum is purely continuous and can be discretely parameterized by a paramter ξ, sometimes referred to as the Bloch parameter or Bloch frequency. Moreover, standard results imply the existence of three separate branches of the spectrum given by $\lambda_j(\xi)$, $j = 1, 2, 3$, which are defined and C^1 for $|\xi| \ll 1$. These branches correspond to the breaking of a Jordan block at $\lambda_j(0) = 0$.

Once the behavior of the localized spectrum of $A[\phi]$ near $\lambda_j(0) = 0$ is understood and once we have translated into the proper coordinate system, we use specral perturbation theory to understand the three branches $\lambda_j(\xi)$ for $|\xi| \ll 1$. In particular, we project onto the three-dimensional total eigenspace associated with our three branches $\lambda_j(\xi)$. This yields a three-dimensional system that, after the proper algebraic manipulation, is *exactly* the three-dimensional Whitham system (up to a harmless shift by the identity). Hence, we establish that the hyperbolicity of the Whitham averaged system is in fact a necessary condition for spectral stability and ellipticity is a sufficient condition for spectral instability. We then show that, in the small amplitude regime, the Whitham system is hyperbolic when the wave number is sufficiently small (i.e. when the wavelength is large) and is elliptic when the wave number is large (i.e. when the wavelength is small). This is a good match with the experimental observations in [12], and it verifies that the elliptic regime found by Maiden and Hoefer in [11] produces spectrally unstable periodic waves.

2 Subharmonic Dynamics of Periodic Waves in Dissipative Systems

First, we consider the linear stability and dynamics of periodic stationary solutions of the Lugiato-Lefever equation (LLE)

$$\psi_t = -i\beta\psi_{xx} - (\delta + i\alpha)\psi + i|\psi|^2\psi + F, \tag{2.1}$$

which is an NLS-type equation with damping, detuning, and driving that is widely used as a model to investigate the dynamical properties of laser fields confined in nonlinear optical resonators [3]. The form of the LLE given in (2.1) is sometimes referred to as the longitudinal/temporal LLE and was derived in the context of dispersive optical ring cavities: [3]. Despite being derived in a different, albeit complementary, physical setting, the two-dimensional transverse LLE is mathematically equivalent (when reduced to one spatial dimension) to the longitudinal/temporal LLE given in (2.1) with $\beta = -1$, i.e. with anamolous dispersion (see paragraph below) [2]. (2.1) may also be derived from the Maxwell-Bloch equations [2]. Moreover, the LLE has been found to be the best framework for the theoretical investigation of Kerr optical frequency comb generation using whispering gallery mode cavities or integrated ring resonators [3], and it provides an outstanding example of phenomena of spontaneous pattern formation [2].

In (2.1), $\psi(x, t)$ is a complex-valued function and represents the electric field envelope, t represents a temporal variable, x can represent a positional variable such as the angle in the circular cavity, $\delta = 1$ represents the damping term, $\alpha \in \mathbb{R}$ represents the detuning parameter, $F \in \mathbb{R}^+$ represents the driving term, and $\beta = \pm 1$ represents the dispersion parameter. If $\beta = 1$, we have normal dispersion, and if $\beta = -1$, we have anomalous dispersion, which is the more physical case.

Our investigation was motivated by previous results [5, 15] establishing spectral stability implies nonlinear asymptotic stability of T-periodic stationary wave solutions when subjected to subharmonic perturbations, i.e. NT-periodic perturbations for some positive integer N. The previous asymptotic stability results establish an exponential decay rate $e^{-\delta_N t}$, where $\delta_N > 0$, and a maximum allowable size of perturbation $\varepsilon_N > 0$. Owing to the methods of proof, i.e. either semigroup theory or center manifold theory, δ_N and ε_N unfortunately tend to 0 as $N \to \infty$, meaning previous results are degenerate in the limit $N = \infty$. Since this limit is physically relevant for the LLE, our goal is to find a rate of decay and allowable size of perturbation that is uniform in N. Our strategy for accomplishing this goal is formally motivated by the study of localized, i.e. integrable on the line, perturbations because such perturbations can formally be seen as the limiting behavior of subharmonic perturbations when N goes to infinity.

Towards that end, [5] uses bifurcation theory to show that there exist periodic stationary wave solutions to the LLE which are spectrally stable when subjected to both subharmonic and localized perturbations. We therefore make an appropriate spectral stability assumption, which is verified to be true in at least the case exhibited in [5]. We use this assumption to obtain asymptotic stability of the C^0 semi-group associated with the linear operator $A[\phi]$, which is obtained from linearizing (2.1) about a spectrally-stable periodic stationary solution ϕ. The linear asymptotic stability holds for localized perturbations and is uniform in N for subharmonic perturbations. Moreover, the uniform asymptotic rate of decay for subharmonic perturbations is *precisely* the polynomial rate of decay coming from localized perturbations.

The tools used to attain linear asymptotic stability are based off a recent general theory for dissipative modulations derived by [8] and are driven by the use of Floquet-Bloch theory and, more specifically, the use of the Bloch Transform (a periodic analogue of the more famous Fourier Transform). Floquet-Bloch theory and the Bloch Transform have previously been heavily developed for localized perturbations. In our work, we make modifications to the relevant theory and transform in order to adapt them to the study of subharmonic perturbations.

This allows us to decompose the semi-group associated with $A[\phi]$ into several pieces where asymptotic decay may be more readily obtained. While carefully keeping track of the dependence on N, our analysis both recovers the previously established degenerate exponential decay result, and it allows us to establish a polynomial decay result which is uniform in N. Moreover, the analysis demonstrates that the uniform subharmonic decay rate is *sharply* controlled by the polynomial decay rate coming from the localized theory. We see a formal convergence of the subharmonic result to the localized result as $N \to \infty$, which was, in fact, the motivation for our approach.

Furthermore, our analysis allows us to describe the long-term dynamics near our background wave ϕ. In particular, the linear evolution of a small subharmonic perturbation $\psi(x,0) = \phi(x) + v_N(x,0)$ will satisfy, for $t \gg 1$,

$$\psi(x,t) = \phi(x) + v_N(x,t) \approx \phi(x) + \phi'(x)\gamma_N(x,t) + \text{“decay”} \approx \phi(x + \gamma_N(x,t)) + \text{“decay,”}$$

where γ_N is a space-time-dependent modulation and where the “decay” is uniform in N. The introduction of space-time-dependent modulations as a mechanism to obtain such uniform in N subharmonic stability results is the “big idea” behind this project. Note that when studying subharmonic perturbations with N-fixed, hence when not seeking uniformity, one only needs to consider time-dependent modulations in order to achieve exponential decay results. As mentioned above, such exponential decay results are degenerate in the limit as $N \to \infty$. Our research therefore shows that, in order to obtain uniform in N stability results, one must allow modulations which depend on both space and time.

The use of space-time-dependent modulations, as compared to the more standard time-dependent modulations, is motivated by the corresponding theory for localized perturbations. In that context, we proved that the linear evolution of a small localized perturbation $\psi(x,0) = \phi(x) + v(x,0)$ similarly satisfies

$$\psi(x,t) = \phi(x) + v(x,t) \approx \phi(x) + \phi'(x)\gamma(x,t) + \text{"decay"} \approx \phi(x+\gamma(x,t)) + \text{"decay,"}$$

where γ is a space-time-dependent modulation. In fact, the dynamics of γ are described by Whitham's theory of wave modulations, which provides justification for Whitham theory for (2.1) at both the spectral level (as in the previously described project) and at the level of linear dynamics. In our work, we further studied the convergence (for appropriate sequences of initial data) of the subharmonic modulation functions γ_N to the modulation function γ coming from the localized theory. Taken together, this demonstrates an intimate connection between subharmonic perturbations, localized perturbations, and Whtitham's theory of wave modulations.

In the case of the LLE, we were unable to push the result to the nonlinear level. This is due to the fact that there is an unavoidable loss of derivatives that occurs in our iteration scheme. In contexts when the PDE has dissipation in the highest-order term, one can show through a "nonlinear damping" result that high Sobolev norms are exponentially slaved to low Sobolev norms, thereby regaining derivatives lost in the iteration. Unfortunately, the dissipation in the LLE occurs in the lowest-order term, leaving no hope of regaining derivatives through a nonlinear damping estimate. We are, however, continuing to study how to work around the low-order dissipation, and this is exciting as it is motivating the development of new techniques (see the section on my current and future research directions below).

Inspired by the ideas presented above, we researched if, in the presence of a nonlinear damping estimate, the methodology we developed for the linear analysis could be used to develop an analogous nonlinear stability result which is uniform in N. This was done in order to to deepen our understanding of the theory and to make progress towards overcoming the difficulties for the LLE outlined above. Towards that end, we consider the nonlinear stability and dynamics of spectrally-stable periodic traveling wave solutions of systems of reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \geq 0,\ u \in \mathbb{R}^n, \tag{2.2}$$

where $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^K-smooth nonlinearity for some $K \geq 3$. Such systems arise naturally in many areas of applied mathematics, and the behavior of such wave train solutions when subject to a variety of classes of perturbations has been studied extensively over the last decade. In particular, localized perturbations have been studied intensively and have been shown to be connected to Whitham's theory of wave modulations [8]. Our goal, as before, is to study the stability of a T-periodic wave solution to (2.2) when subjected to subharmonic, i.e. NT-periodic, perturbations and thereby establish a stability result which is uniform in N at the nonlinear level.

Following the linear analysis outlined in the case of the LLE above (see also [7]), we find that the linear analysis predicts that the evolution of a small subharmonic perturbation $u(x,0) = \phi(x) + v_N(x,0)$ will satisfy, for $t \gg 1$

$$u(x,t) = \phi(x) + v_N(x,t) \approx \phi(x) + \phi'(x)\gamma_N(x,t) + \text{"decay"} \approx \phi(x+\gamma_N(x,t)) + \text{"decay,"}$$

where γ_N is a space-time-dependent modulation and where the "decay" is uniform in N and is precisely that given by the localized theory. We then introduce a decomposition of the nonlinear perturbation of the background wave ϕ which accounts for the space-time-dependent modulation predicted by the linear theory. We find that the nonlinear perturbation and space-time-dependent modulation must satisfy a system of implicit integral equations. Using the fact that dissipation comes from the term with the highest number of derivatives, we are able to derive a result commonly referred to as nonlinear damping that allows us to close a standard nonlinear iteration scheme.

It is worth recalling that we must do all of the above analysis while keeping track of the dependence on N so that the nonlinear result is indeed uniform in N. Moreover, we note that the space-time dependent modulation is slightly more complicated for subharmonic perturbations than it is for localized perturbations, in part owing to the more discrete nature of the subharmonic Bloch transform. Lastly, the uniform subharmonic nonlinear rate of decay is again precisely that given by the localized nonlinear rate of decay.

Current and Future Research Directions

I plan on continuing to study the existence, stability, and dynamics of solutions of nonlinear PDEs arising in physical systems. Specifically, the research questions that I currently intend to pursue may be summarized as follows:

1. **Three-Dimensional Viscous Fluid Conduits**

   Along with Prof. Mathew Johnson and Prof. Mark Hoefer[2], I am currently seeking to derive a model for viscous fluid conduits (like those described in Section 1) that is fully three-dimensional. Ideally this model will be able to explain the robustness of symmetrical patterns observed in physical experiments, and we hope it will be able to explain asymmetrical patterns that are experimentally observable but are not predicted by the current two-dimensional model. The derivation of such a model should be achieved by performing an appropriate multiple scales analysis on the associated boundary value problem for the Navier-Stokes equation describing the experimental setup. While this has been achieved for two-dimensional viscous fluid conduits, i.e. those assuming a rotational symmetry, the new challenge here will be to break the rotational symmetry assumption by adding in the Coriolis effect to the Navier-Stokes equations. This constitutes the rigorous derivation of a new mathematical model, and this new model will be numerically explored and experimentally verified. These numerical and experimental investigations are quite amenable to undergraduate- and Master's-level research programs.

2. **Nonlinear Stability of Spectrally Stable Periodic Lugiato-Lefever Waves**

   Along with Prof. Mariana Haragus[3], Prof. Johnson, and Dr. Björn de Rijk[4], I am currently researching how to extend the uniform subharmonic linear asymptotic stability obtained for the Lugiato-Lefever Equation (LLE) in Section 2 to full nonlinear stability. As described in Section 2, the lack of highest-order dissipation present in the LLE prevents the use of the more traditional approach known as "nonlinear damping." However, we hope to attain full nonlinear stability by adapting recent techniques from the paper [4] to circumvent the lack of highest-order dissipation present in the LLE. This adaptation is complicated by the fact

[2]University of Colorado Boulder
[3]FEMTO-ST institute, Univ. Bourgogne-Franche Comté
[4]Universität Stuttgart

that [4] uses pointwise estimates while our linear asymptotic stability analysis is presented in terms of $L^2(\mathbb{R})$. Nevertheless, we are confident that we can attain a nonlinear subharmonic stability result that is uniform in N, something that is very important to the physicists who conduct experiments modeled by the LLE.

3. **Nonlinear Stability of Viscous Conservation Laws**

   Jointly with Prof. Johnson, I am currently researching how to attain uniform stability to subharmonic perturbations at the nonlinear level in the general case of viscous conservation laws. The analysis in this case is complicated by the fact that there are multiple eigenvalues at the origin (something that was not present in the case of the Lugiato-Lefever Equation or the system of reaction-diffusion equations). This leads to the presence of additional modulations, in particular modulations in the mass and wavespeed of the background wave, that are not present in the aforementioned equations. The presence of these additional modulations complicates the analysis as we must show that the phase modulation coming from translation invariance still dominates the dynamics. Furthermore, we no longer expect linear stability (owing to the breaking of a Jordan block at the origin), but we believe that nonlinear stability is still possible, as demonstrated in the case of localized perturbations in [8].

4. **Further Justification of Whitham Theory**

   I am interested in rigorously justifying Whitham's theory of wave modulations for a generalized nonlinear wave equation (in particular, a generalized Whitham equation) with general pseudo-differential operator and general nonlinearity. This equation can be written as

   $$u_t + f(u)_x + \mathcal{K} * u_x = 0, \quad \text{where} \quad \widehat{\mathcal{K} * g(x)} = \frac{\Omega(k)}{k}\widehat{g}(k),$$

   where $f(u)$ and $\Omega(k)$ are the problem-dependent nonlinear hydrodynamic flux and linear dispersion relation, respectively. Whitham theory has recently been used to numerically study this equation in [1]. Building on my experience of rigorously justifying Whitham theory for the conduit equation [9], my plan is to extend previous rigorous justification of Whitham theory for a generalized nonlinear wave equation with a quadratic nonlinearity to that of a general nonlinearity, which is a nontrivial extension. (If possible, it would be interesting to try to fully justify Whitham theory in general.)

5. **Rigorous and Numerical Bifurcation of Periodic Lugiato-Lefever Waves**

   I am interested in numerically bifurcating from spectrally-stable waves that are shown to exist in [5, 6], to see if there are more stationary periodic wave solutions to the LLE that satisfy the assumptions we make in [7]. A project of this form would be accessible to sufficiently advanced undergraduate- and Master's-level research students.

   Furthermore, I am interested in rigorously bifurcating from constant solutions in the 2-D LLE in order to justify known pattern formations. Such rigorous bifucation would also be suitable to undergraduate- and Master's-level research projects.

More information on my research, including all of my publications, can be found on my webpage at http://people.ku.edu/~w128p157/

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