# Modulational and Subharmonic Dynamics of Periodic Waves



Wesley Perkins

B.A. Mathematics, Lyon College, 2015

M.A. Mathematics, University of Kansas, 2017

Submitted to the graduate degree program in Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Committee members

Mathew Johnson, Chair

Geng Chen

Dionyssis Mantzavinos

Milena Stanislavova

Mike Wuthrich, External Reviewer

Date defended: May 4, 2021

The Dissertation Committee for Wesley Perkins certifies
that this is the approved version of the following dissertation :

Modulational and Subharmonic Dynamics of Periodic Waves

____________________________________

Mathew Johnson, Chair

Date approved: May 4, 2021

# Abstract


The overarching theme of this dissertation is to investigate the stability and dynamics of spatially-periodic travelling and stationary wave solutions to partial differential equations (PDEs) arising from physical applications.

First we study the modulational dynamics of interfacial waves rising buoyantly along a conduit of a viscous liquid. Formally the behavior of modulated periodic waves on large space and time scales may be described through the use of Whitham modulation theory. The application of Whitham theory, however, is based on formal asymptotic (WKB) methods, thus removing a layer of rigor that would otherwise support their predictions. In this first study, we aim at rigorously verifying the predictions of the Whitham theory, as it pertains to the modulational stability of periodic waves, in the context of the so-called conduit equation, a nonlinear dispersive PDE governing the evolution of the circular interface separating a light, viscous fluid rising buoyantly through a heavy, more viscous, miscible fluid at small Reynolds numbers. Using rigorous spectral perturbation theory, we connect the predictions of Whitham theory to the rigorous spectral (in particular, modulational) stability of the underlying wave trains.

We then study the linear dynamics of spectrally stable $T$-periodic stationary solutions of the Lugiato-Lefever equation (LLE), a damped nonlinear Schrödinger equation with forcing that arises in nonlinear optics. It is known that such $T$-periodic solutions are nonlinearly stable to $NT$-periodic, i.e., subharmonic, perturbations for each $N \in \mathbb{N}$ with exponential decay rates of the form $e^{-\delta_N t}$. However, both the exponential rates of decay $\delta_N$ and the allowable size of initial perturbations tend to 0 as $N \to \infty$ so that this result is non-uniform in $N$ and is, in fact, empty in the limit $N = \infty$. We introduce a methodology in the context of the LLE by which a uniform stability result for subharmonic perturbations may be achieved at the linear level. The obtained uniform decay rates are shown to agree precisely with the polynomial decay rates of localized, i.e., integrable on the

real line, perturbations of such spectrally stable periodic solutions of the LLE. A key component of the proofs in this study is the introduction of space-time dependent modulations, which, as we show, allows us to justify Whitham theory at the level of linear dynamics for the LLE. This work both unifies and expands on several existing works in the literature concerning the stability and dynamics of such waves, and sets forth a general methodology for studying such problems in other contexts.

Interestingly, we were unable to push the above analysis to the nonlinear level due to an unavoidable loss of derivatives that occurs in our iteration scheme. If the PDE has dissipation in the highest-order term, one can regain these lost derivatives through a technique known as "nonlinear damping." Unfortunately, the dissipation in the LLE occurs in the lowest-order term, leaving no hope of regaining derivatives through a nonlinear damping estimate. As a proof of concept, we explore if, in the presence of a nonlinear damping estimate, the methodology we developed for the linear analysis could be used to develop an analogous nonlinear subharmonic stability result which is uniform in $N$. To that end, we investigated the stability and nonlinear dynamics of spectrally-stable wave trains in reaction-diffusion systems. Using the nonlinear damping estimate present in reaction-diffusion systems, we were able to successfully introduce a methodology by which a stability result for subharmonic perturbations which is uniform in $N$ may be achieved at the *nonlinear* level.

This proof of concept therefore motivates the idea that methodologies for overcoming the loss of regularity that occurs in *localized* nonlinear iteration schemes can be modified to similarly overcome the loss of regularity that occurs when establishing nonlinear subharmonic stability results that are uniform in the period of perturbation. Consequently, in the final chapter of the dissertation we develop a new methodology that allows us to circumvent the loss of regularity in the case of localized perturbations and in the presence of weak damping, i.e., in the absence of a nonlinear damping estimate. We return to our study of the Lugiato-Lefever equation and consider the nonlinear stability of spectrally stable periodic stationary solutions of the LLE. In our first study of the LLE, we used a delicate decomposition of the associated linearized solution operator to obtain linear stability results to localized perturbations with polynomial rates of decay to a spatio-temporal phase modulation of the underlying wave. In this study, we present a new nonlinear iteration scheme in which the aforementioned loss of derivatives is compensated through a coupling to a separate "unmodulated" iteration scheme in which derivatives are not lost, yet where perturbations decay too slow to close an independent iteration scheme. Our work establishes the nonlinear stability of spectrally stable periodic stationary solutions of the LLE to localized perturbations with precisely the same polynomial decay rates predicted from the linear theory. Moreover, as our study of reaction-diffusion systems showed, it motivates the methodology by which we may obtain a nonlinear subharmonic stability result for the LLE that is uniform in the period of perturbation, a result that is currently under investigation by the author et al.

# Acknowledgements

As I have neared the end of my time as a graduate student at the University of Kansas, I have been reflecting a lot on my experiences over the past six years. There are a number of individuals who have been instrumental to my growth and development during my time at KU, and I would like to acknowledge and express my gratitude towards these individuals.

First and foremost, I am incredibly grateful to the LORD God. I would have given up on my pursuit of a doctoral degree without Him, yet He did more than help me decide to stay in graduate school. Indeed, He has given me a passion and vision for both teaching and research. He has taught me much, brought healing in my life, shaped me through my experiences and relationships, and so much more. Thank You.

I feel a deep sense of gratitude towards my Ph.D. advisor and mentor, Professor Mathew Johnson. I could not have asked for a better advisor. He has been relentlessly supportive in all of my academic pursuits. He has known when to spur me onward and when to slow down and talk about life, which has been instrumental in helping me navigate a healthy work-life balance. I am grateful for his sponsorship of me as a GRA over several summers and for the Spring 2020 semester, in particular; without this financial support, this dissertation would not exist. He was instrumental in a successful (albeit, long and grueling) academic job search. Most importantly, I am thankful for his friendship and camaraderie over the past four years.

I would like to thank Professors Geng Chen, Dionyssis Mantzavinos, Milena Stanislavova, and Mike Wuthrich for serving on the committee for both my oral comprehensive exam and my final defense. I would further like to thank Professors Dionyssis Mantzavinos and Milena Stanislavova for serving on the committee for my preliminary exam.

I am grateful for the numerous conversations with my research collaborators over the years. I would like to thank Professor Mark Hoefer for several helpful and ongoing discussions regarding the dynamics of viscous fluid conduits. I would further like to thank Professors Mariana Haragus and Björn de Rijk for their significant contributions to results presented in this dissertation and for their numerous conversations regarding the dynamics of the Lugiato-Lefever equation. I am also grateful to Professor Guido Schneider for initial discussions regarding Corollary 4.1.9.

I wish to thank the KU Department of Mathematics for supporting me as a GTA throughout my time here. I also want to thank the many instructors and course coordinators who have invested in me. They have been instrumental in my development as a teacher and professional mathematician. I am thankful for the team of GTAs/lecturers that I coordinated for Math 104 in the Fall 2020 semester. They were a great team to work with! I would also like to express my gratitude to all of the wonderful students that I have had throughout my time at KU. Teaching them was one of the greatest highlights of my graduate school career.

I am sincerely appreciative of my dear friends and colleagues Harry Gaebler, Brad Isom, Cassidy Krause, Heather Fielding, Tommy Gebhardt, and Amanda Wilkens. Their camaraderie, friendship, and support over the past six years (and the first few semesters, in particular) is worth more than words can say. I hope we can continue to reminisce about apples and smoking ghosts for many years to come. I also wish to express gratitude to Brad Isom and Harry Gaebler for their many helpful orienting discussions as I was developing the Bloch transform for subharmonic functions.

I am incredibly grateful for my best friend and former roommate Jayden Garetson. I wouldn't have survived the first few years of graduate school without him, and I certainly wouldn't be the man I am today without his constant friendship, support, and willingness to talk about anything. I would further like to express my gratitude towards the many friends I have made through First Southern Baptist Church, KU Christian Challenge, Rock Hill Church, and my Common Life Community. The friendships I made in these communities were absolutely formational during my time at KU.

Finally, words are insufficient to express the gratitude that I have towards my family. My parents, Robert and Daphne, my brother, Jacob, my sister-in-law, Morgan, my sister, Nikkolette, my grandparents, especially Betty Joyce, and the rest of my extended family have all been so loving and supportive of me. My parents provided me with an incredible foundation growing up, without which my success in education would not have been possible, and they have continued to support me in all of my highs and lows. Thank you both! :) I am also thankful for my incredible siblings and their many conversations and virtual/in-person hangouts over the years, which have been a constant source of joy. Thank you! :)

# Contents

# List of Figures

# Chapter 1

# Background and Overview

**Abstract**

The goal of this first chapter is to introduce the notions of stability and classes of perturbations that we will study in this dissertation, as well as to provide some common theoretical framework necessary to study the notions of stability and perturbations that we introduce.

## 1.1 Introduction

This dissertation focuses on the stability and dynamics of nonlinear wave solutions to partial differential equations (PDEs) arising from physical applications. In particular, we will study nonlinear waves and coherent structures that are motivated by experimental or numerical observations. These physically observable patterns can often be studied through a PDE that models the physical system. Several questions naturally arise when using a PDE to study observed patterns:

- Does an idealized version of the observed pattern *exist* as a solution to the PDE?
- Is such a solution *stable*, i.e., is it able to persist in the presence of a small perturbation?
- What are the *local dynamics* (i.e., the dynamics near such a solution) within the PDE as time evolves?
- Can the PDE predict *new observations*?
- Can it explain *why* we see some patterns and not others?

The first three questions endeavor to determine, among other things, whether or not the PDE is a good model for the physical system, and their answers can gauge the strengths and weaknesses of the PDE and inform what (if anything) needs to change in the model. The last two questions seek to understand what the PDE can teach us about the underlying physical phenomena, and the answers to these questions can motivate new experimental and numerical research.

The stability of a particular class of solutions is of fundamental practical importance as solutions which are unstable do not naturally arise in applications except possibly as transient phenomena. In particular, understanding why some patterns are stable and others are unstable can motivate methods of stabilizing unstable patterns via the addition of controls to the physical system. The ability to stabilize unstable patterns is key in many applications where there exist unstable, yet desirable, structures.

In the second bullet point above, we motivated the formal notion of stability as the persistence of a wave in the presence of a small perturbation. To put this in a more mathematical framework, consider the PDE

$$u_t = F(u) \tag{1.1.1}$$

posed on a Banach space $X$. Suppose $F : X \longrightarrow X$ is some generic nonlinear operator, and let $\phi \in X$ be a stationary wave solution[1] to (1.1.1), i.e.,

$$\phi_t = 0 \Longrightarrow F(\phi) = 0.$$

At this point, we introduce the new formal definition of stability as follows.

> If $u(t)$ is initially close to $\phi$, i.e., if $u(0) = u_0$ is close to $\phi$ (in an appropriate sense of "closeness"), then $\phi$ is *stable* if $u(t)$ remains close to $\phi$ (again, in an appropriate sense) for all time $t > 0$.

In order to understand what we mean by "closeness," we need to introduce tools that will allow us to measure the distance between $u_0$ and $\phi$ and between $u(t)$ and $\phi$ for $t > 0$, respectively. To

[1]Note that if we wish to study a traveling wave solution $\phi$, then we simply shift (1.1.1) to a traveling coordinate frame, where $\phi$ is a stationary solution.

that end, note that we may write

$$u(t) = \phi + v(t),$$

where $v(t) := u(t) - \phi$ and where $v(0) = v_0 := u_0 - \phi$ is assumed to be "small." Consequently, in order for $\phi$ to be stable, we need $v(t)$ to be "small" so that $u(t)$ is "close" to $\phi$ for $t > 0$. This allows us to interpret $u(t)$ as a perturbation of $\phi$, where $v(t)$ is our initially small perturbation.

The benefit of viewing $u(t)$ as a perturbation of $\phi$ is that there are often many classes of perturbations worth studying. Furthermore, these classes of perturbations motivate our notion of "closeness." More explicitly, we may wish to study the stability of $\phi$ with respect to perturbations $v(t) \in V$ for some Banach space $V$, where $V$ may or may not be related to $X$. The topology of $V$ provides us with the ability to measure the relative size of an object; for instance, we can measure the size of $v(t)$ with the $V$-norm, $\|\cdot\|_V$. We therefore conclude that stability can formally be defined as follows.

> If $v(t)$ *starts* small, i.e., if $\|v_0\|_V \ll 1$, then $\phi$ is stable if $v(t)$ *stays* small, i.e., if $\|v(t)\|_V \ll 1$ for $t > 0$."[2]

Note that this more mathematical formulation of stability is consistent with our original notion that $\phi$ is stable if it persists in the presence of initially small perturbations.

While there are all types of nonlinear waves and coherent structures that are studied using this formal mathematical framework, there are two broad categories that are of significant interest; namely, asymptotically constant structures (e.g., pulses and fronts) and spatially-periodic structures. Each broad category comes with its own associated classes of initial perturbations that are physically relevant. This dissertation will focus on the study of spatially-periodic structures, and we now introduce the pertinent ambient space $X$ and perturbation space(s) $V$ that are used to study the stability of spatially-periodic structures.

Consider a $T$-periodic stationary wave solution $\phi$ to (1.1.1) posed on $X$. Since $\phi$ is $T$-periodic,

[2]Generically, the notion for which we measure the smallness of $v_0$ need not be the same as the notion for which we measure the smallness of $v(t)$. Moreover, these notions of sizes need not be norms or metrics, but can be seminorms or pseudometrics. (In fact, pseudometrics are relevant in many physical applications, as we will see in later chapters.) For the purposes of this introduction, we stick with norms.

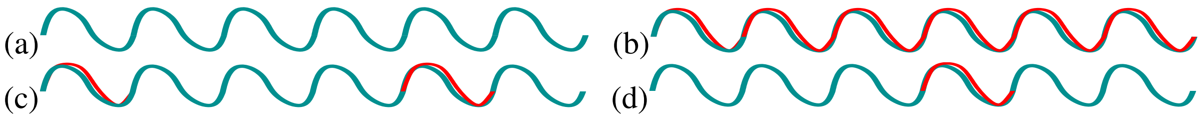



Figure 1.1: (a) A representative example of a stationary $T$-periodic wave $\phi \in L^p_{\text{per}}(0,T)$. (b) An example of when the background wave $\phi$ is subjected to a small subharmonic perturbation $v_0$ (here in red). Observe that the perturbation has the same period as the background wave so that it is $NT$-periodic with $N = 1$, i.e., it is a small co-periodic perturbation. (c) Another example of when the background wave $\phi$ is subjected to a small subharmonic perturbation $v_0$. Here, the perturbation has 4 times period as the background wave so that it is $NT$-periodic with $N = 4$. (d) An example of when the background wave $\phi$ is subjected to a small localized perturbation $v_0$.

one natural choice for the space $X$ would be

$$X = L^p_{\text{per}}(0,T) := \{f \in L^p(0,T) \mid f(x+T) = f(x) \text{ for a.e. } x\},$$

where $p \in [1,\infty]$. In other words, $X$ represents the space of $T$-periodic functions that are $p$-integrable over one period. For example, see Figure 1.1(a). In order to study the stability of $\phi$, we need to identify the space $V$ of admissible initial perturbations $v_0$. When studying periodic solutions to PDEs of the form (1.1.1), there are two imporant and widely studied classes of perturbations that arise naturally in applications. One class of perturbations is referred to as the class of *subharmonic* perturbations, i.e., $NT$-periodic perturbations[3] for some positive integer $N$. In this case, we have that

$$V = L^p_{\text{per}}(0,NT)$$

for some $N \in \mathbb{N}$, e.g., see Figure 1.1(b)-(c). The other class of perturbations is referred to as the class of *localized* perturbations, i.e., perturbations which are integrable on the real line. In this case, we have that

$$V = L^p(\mathbb{R}),$$

e.g., see Figure 1.1(d). We note that, while most of our results can be extended to the $X$ and $V$ listed above for general $p$, we choose to work in the Hilbertian framework when $p = 2$.

[3]Note that the special case of $N = 1$ is often referred to as *co-periodic* perturbations, in which case $V = X$. For example, see Figure 1.1(b).

Observe that localized perturbations can *formally* be understood as the limit of subharmonic perturbations as $N \to \infty$. Indeed, notice that as $N$ increases without bound, functions in $L^p_{\text{per}}(0, NT)$ must become increasingly localized in order for the integral over one period to converge.[4] This observation will be critical in our study of subharmonic perturbations.

Our goal for the rest of this chapter is as follows. First, we make our formal notion of stability rigorous for a general $X$ and $V$. Then we introduce the main tools needed to study the stability of spatially-periodic structures $\phi \in X = L^2_{\text{per}}(0, T)$ in the presence of both localized perturbations, i.e., $V = L^2(\mathbb{R})$, and subharmonic perturbations, i.e., $V = L^2_{\text{per}}(0, NT)$.

## 1.2 Preliminaries

Recall we are studying the PDE (1.1.1)

$$u_t = F(u)$$

posed on $X$, where $F : X \to X$ is some generic nonlinear operator. Further recall that $\phi \in X$ is a stationary wave solution to (1.1.1), i.e.,

$$\phi_t = 0 \implies F(\phi) = 0.$$

As motivated above, we wish to study the stability of $\phi$ with respect to perturbations $v(t) \in V$, where $V$ and $X$ generically have no relationship to each other. In order to account for the possible mismatch of spaces, we consider the perturbed solution

$$u(t) = \phi + v(t), \quad u_0 = \phi + v_0,$$

[4]Alternatively, if we consider Figure 1.1, we can see that the subharmonic perturbations necessarily become more sparse, i.e., they become more localized, as $N$ increases without bound.

where $u(0) = u_0$ and $v(0) = v_0$. Substituting this into (1.1.1) gives rise to the PDE describing the evolution of the perturbation, i.e., the perturbation evolution equation. In other words, we see that $v(t)$ satisfies the IVP

$$v_t = F(\phi + v), \quad v(0) = v_0 \tag{1.2.1}$$

posed on $V$.[5] We assume that (1.2.1) is locally well-posed (see Definition 1.2.1) on a dense subspace $W$ that is continuously embedded in $V$.[6]

**Definition 1.2.1** (Local Well-Posedness)**.** We say that (1.2.1) is *locally well-posed* on $W \subseteq V$ if for all $v_0 \in W$, there exists a time $\tau = \tau(v_0) > 0$ such that there exists a unique solution $v(t) \in W$ of (1.2.1) for $t \in [0, \tau)$ with $v(0) = v_0$. Moreover, $\tau < \infty$ if and only if

$$\lim_{t \nearrow \tau} \|v(t)\|_V = \infty.$$

This last condition is sometimes referred to as the *blow-up alternative*.

Since $v = 0$ is clearly an equilibrium solution to (1.2.1) and since

$$\|u(t) - \phi\|_V \ll 1 \iff \|v(t) - 0\|_V \ll 1,$$

we have that the stability of $\phi \in X$ as a solution to (1.1.1) is equivalent to the stability of $0 \in V$ as a solution to (1.2.1). Therefore, we say that $\phi$ is a nonlinearly stable solution of (1.1.1) if the following definition holds.

**Definition 1.2.2** (Nonlinear Stability)**.** Let $\phi \in X$ be a stationary solution to (1.1.1). Then, $\phi$ is said to be *nonlinearly stable* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $v_0 \in W \subseteq V$ and $\|v_0\|_V < \delta$, then the unique solution $v(t)$ of (1.2.1) with initial data $v(0) = v_0$ exists for all

[5]As we will see below, when $\phi$ is a stationay wave solution to (1.1.1), $F(\phi + v) = O(\|v\|_V)$. Thus, it is reasonable to expect that $F(\phi + \cdot) : V \to V$.

[6]In general, we do not need (1.2.1) to be locally well-posed on a dense subspace of $V$. In fact, we only need (1.2.1) to be locally well-posed in an open neighborhood of the origin.

$t > 0$ and satisfies

$$\|v(t)\|_V \leq \varepsilon.$$

Otherwise, we say that $\phi$ is *nonlinearly unstable*. If, in addition, we have that $\|v(t)\|_V \to 0$ as $t \to \infty$, then we say that $\phi$ is *nonlinearly asymptotically stable*.

At this point, it is natural to ask how we prove that $\phi$ is a nonlinearly stable solution to (1.1.1). Well, recall (1.2.1)

$$v_t = F(\phi + v)$$

and note that we can write

$$F(\phi + v) = F(\phi) + DF(\phi)v + \underbrace{[F(\phi + v) - F(\phi) - DF(\phi)v]}_{O(\|v\|_V^2)},$$

where here $\mathcal{L}[\phi] := DF(\phi)$ is the Fréchet derivative of $F$ evaluated at $\phi$ considered as a densely-defined operator from $V$ into itself, i.e., $\mathcal{L}[\phi] : D(\mathcal{L}[\phi]) \subseteq V \to V$. In order to make sure the problem remains well-posed, we assume that the domain of $\mathcal{L}[\phi]$ satisfies

$$D(\mathcal{L}[\phi]) \subseteq W \subseteq V,$$

where each subspace is dense and continuously embedded. Recalling that $F(\phi) = 0$, we find that

$$v_t = \mathcal{L}[\phi]v + \mathcal{N}(v), \tag{1.2.2}$$

where $\mathcal{N}(v) = F(\phi + v) - DF(\phi)v = O(\|v\|_V^2)$.

Note that (1.2.2) is clearly equivalent to (1.2.1). However, splitting $F$ into its linear and nonlinear components is beneficial since, assuming that $\mathcal{L}[\phi]$ generates a $C^0$-semigroup, we can use Duhamel's formula to rewrite (1.2.2) as the implicit integral equation

$$v(t) = e^{\mathcal{L}[\phi]t}v_0 + \int_0^t e^{\mathcal{L}[\phi](t-s)} \underbrace{\mathcal{N}(v(s))}_{O(\|v(s)\|_V^2)} \, ds, \tag{1.2.3}$$

where $e^{\mathcal{L}[\phi]t}v_0$ is the unique solution to the linear IVP

$$v_t = \mathcal{L}[\phi]v, \quad v(0) = v_0 \tag{1.2.4}$$

posed on $V$. In general, it proves to be quite challenging to study the implicit integral equation (1.2.3) straight out of the gate. Instead, we often approach this problem by observing that *if* $\|v(t)\|_V$ is small, then $\|v(t)\|_V^2$ would be much smaller than $\|v(t)\|_V$. Therefore, it stands to reason that solutions to (1.2.1) will be well approximated by solutions to (1.2.4). This motivates the following notion of stability.

**Definition 1.2.3** (Linear Stability)**.** Let $\phi \in X$ be a stationary solution of (1.1.1). Then $\phi$ is said to be *linearly stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $v_0 \in V$ and $\|v_0\|_V < \delta$, then the unique solution $v(t)$ of the linearized problem (1.2.4) with initial data $v(0) = v_0$ exists for all $t > 0$ and satisfies

$$\|v(t)\|_V \leq \varepsilon.$$

Otherwise, we say that $\phi$ is *linearly unstable*. If, in addition, we have that $\|v(t)\|_V \to 0$ as $t \to \infty$, then we say that $\phi$ is *linearly asymptotically stable*.[7]

While proving the linear stability of $\phi$ is generally easier than proving the nonlinear stability of $\phi$, it is still often a challenging problem. So how do we study the linear stability of $\phi$? Well, we know that (1.2.4) has the unique solution

$$v(t) = e^{\mathcal{L}[\phi]t}v_0,$$

[7]Note that for the notion of linear stability, we dropped the space $W$, where the *nonlinear* problem was locally well-posed.

where $e^{\mathcal{L}[\phi]t}$ is referred to as the linear solution operator. Consequently, the linear stability of $\phi$ often comes down to establishing bounds on the linear solution operator.

Standard theory from the study of ordinary differential equations (ODEs) actually gives us some insight into how to establish these bounds. Indeed, we know from ODE theory that the eigenvalues, i.e., the spectrum, of the linear operator completely determine the bounds on the linear solution operator. While this is a much more delicate matter when studying PDEs, the ODE theory still provides motivation for the ability to use spectral information to study the linear solution operator. As such, we make the following definitions.

**Definition 1.2.4.** Suppose $L$ is a closed linear operator with dense domain $D(L) \subseteq Z$ for some Banach Space $Z$. Then the *resolvent set* of $L$ acting on $Z$ is given by

$$\rho_Z(L) := \{\lambda \in \mathbb{C} \mid (L - \lambda I)^{-1} \text{ is a bounded linear operator}\}.$$

Furthermore, the *spectrum* of $L$ acting on $Z$ is given by

$$\sigma_Z(L) := \mathbb{C} \setminus \rho_Z(L).$$

In many applications, it is useful to further decompose the spectrum of $L$ as follows.

**Definition 1.2.5.** Suppose $L$ is a closed linear operator with dense domain $D(L) \subseteq Z$ for some Banach Space $Z$. We say that $\lambda \in \sigma_Z(L)$ is an *eigenvalue* of $L$ (in $Z$) if

$$\ker(L - \lambda I) \neq \{0\},$$

i.e., if there exists a $z \in Z \setminus \{0\}$ such that $Lz = \lambda z$. Then, the *point spectrum* of $L$ acting on $Z$ is given by

$$\sigma_{\text{pt},Z}(L) := \{\lambda \in \sigma_Z(L) \mid \lambda \text{ is an isolated eigenvalue of finite multiplicity}\},$$

and the *essential spectrum* of $L$ acting on $Z$ is given by

$$\sigma_{\text{ess},Z}(L) := \sigma_Z(L) \setminus \sigma_{\text{pt},Z}(L).$$

Recalling from ODE theory that linear stability is only possible when all of the eigenvalues of $L$ have nonpositive real parts, we define another notion of stability.

**Definition 1.2.6** (Spectral Stability)**.** Let $\phi \in X$ be a stationary solution of (1.1.1). Then $\phi$ is said to be *spectrally stable* if the spectrum of the linear operator $\mathcal{L}[\phi]$, coming from the linearized problem (1.2.4), acting on $V$ lies in the closed left half-plane,[8] i.e., if

$$\sigma_V(\mathcal{L}[\phi]) \subseteq \{\lambda \in \mathbb{C} \mid \Re(\lambda) \leq 0\}.$$

Otherwise, we say that $\phi$ is *spectrally unstable.*

In an ideal situation, we can use the spectral information to climb up the ladder of stability and establish nonlinear stability. In other words, if we can establish the spectral stability of $\phi$, then we try to use the spectral information to establish bounds on the linear solution operator $e^{\mathcal{L}[\phi]t}$. These bounds hopefully allow us to conclude linear stability, at which point we try to use some sort of nonlinear iteration scheme on the implicit integral equation (1.2.3) in order to conclude nonlinear stability. While the strategy sounds straightforward enough, it can be quite delicate to execute in practice.

Again, it is natural to ask how we study the spectral stability of $\phi$. In general, when $X \neq V$, studying the spectral stability is very tricky due to the mismatch of spaces where $\phi$ and $v$ live. Recall, however, that we will be specifically focusing on the stability of spatially-periodic waves, so we now turn our attention to the specific choices of $X$ and $V$ given above.

In the case when $X = L^2_{\text{per}}(0, T)$ and $V = L^2(\mathbb{R})$, there is a well-developed theory for studying the spectral (in)stability of $\phi$, which we will explore in the next section. Moreover, there is a

[8]This can equivalently be expressed as $\Re(\sigma_V(\mathcal{L}[\phi])) \leq 0$.

well-developed transform that can be used to translate the spectral information into bounds on the linear solution operator. Hence, this transform is often used to establish the linear (in)stability of $\phi$, at which point a nonlinear iteration scheme may be used to establish the nonlinear (in)stability of $\phi$.

In the case when $X = L^2_{\text{per}}(0,T)$ and $V = L^2_{\text{per}}(0,NT)$, we *introduce* a new, but related, theory to aid in the study of the spectrum of $\mathcal{L}[\phi]$. Moreover, we will introduce a new, but related, transform, which in turn helps us study the linear and nonlinear (in)stability of $\phi$.

## 1.3 Floquet-Bloch Theory

We now assume that the linearization of (1.1.1) about a $T$-periodic stationary wave solution $\phi \in L^2_{\text{per}}(0,T)$ gives rise to the linear problem

$$v_t = \mathcal{A}[\phi]v,$$

posed on either $V = L^2(\mathbb{R})$ or $L^2_{\text{per}}(0,NT)$ for some positive integer $N$. We further assume that $\mathcal{A}[\phi]$ is a degree $n$ differential operator with $T$-periodic coefficients,[9] and it is assumed that $\mathcal{A}[\phi]$ is closed when acting on either $V = L^2(\mathbb{R})$ or $L^2_{\text{per}}(0,NT)$ with dense domain $D(\mathcal{A}[\phi]) = H^s(\mathbb{R})$ or $H^s_{\text{per}}(0,NT)$, respectively, for some $s \geq 1$. In order to study the spectral stability of $\phi$, we need to study the spectral problem

$$\mathcal{A}[\phi]v = \lambda v \tag{1.3.1}$$

and note that, since (1.3.1) is an ODE, we can rewrite it as a linear homogeneous differential equation

$$Y' = A(x;\lambda)Y \tag{1.3.2}$$

for some $n \times n$ matrix $A(x;\lambda)$ of complex continuous functions such that $A(x+T;\lambda) = A(x;\lambda)$. By standard Floquet Theory (for details, see [40]), we know that any fundamental matrix solution

[9]In [21], we study the Floquet theory for an operator $\mathcal{A}$ that is a general $m \times m$-matrix differential operator. For extensions beyond the case where $\mathcal{A}$ is a differential operator, please see [20, 28].

of (1.3.2) must be of the form

$$\Phi(x;\lambda) = P(x;\lambda)e^{B(\lambda)x},$$

where $P(x+T;\lambda) = P(x;\lambda) \in \mathbb{C}^{n\times n}$ and where $B(\lambda) \in \mathbb{C}^{n\times n}$ is some constant (in $x$) matrix. It then follows that any solution to (1.3.1) must be of the form

$$v(x;\lambda) = e^{\mu(\lambda)x} p(x;\lambda),$$

where $p(x+T;\lambda) = p(x;\lambda)$ and $\mu(\lambda)$ is an eigenvalue of $B(\lambda)$. Observe that if $\Re(\mu(\lambda)) > 0$, then

$$v(x;\lambda) \xrightarrow{x\to-\infty} 0, \quad v(x;\lambda) \xrightarrow{x\to\infty} \infty$$

at exponential rates. Similarly, if $\Re(\mu(\lambda)) < 0$, then

$$v(x;\lambda) \xrightarrow{x\to-\infty} \infty, \quad v(x;\lambda) \xrightarrow{x\to\infty} 0$$

at exponential rates. In either case, we have that $v(x;\lambda) \notin V$ for either $V = L^2(\mathbb{R})$ or $L^2_{\text{per}}(0, NT)$. Consequently, $v(x;\lambda)$ is at best a bounded solution of the form

$$v(x;\lambda,\xi) = e^{i\xi x} w(x;\lambda,\xi), \tag{1.3.3}$$

for some $\xi \in [-\pi/T, \pi/T)$ and where $w \in L^2_{\text{per}}(0,T)$. At this point, we now individually study the implications of this analysis in the localized case, i.e., when $V = L^2(\mathbb{R})$, and in the subharmonic case, i.e., when $V = L^2_{\text{per}}(0, NT)$.

### 1.3.1 Floquet-Bloch Theory for Localized Perturbations

To begin our study of the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$, we have shown that standard results from Floquet theory imply that non-trivial solutions of the spectral problem (1.3.1)

$$\mathcal{A}[\phi]v = \lambda v$$

cannot be integrable[10] on $\mathbb{R}$ and that, at best they are bounded functions on the line: see, for example, [40, 55]. In particular, the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$ can contain no eigenvalues, and hence must be entirely essential. Further, we have shown that any bounded solution of (1.3.1) must be of the form (1.3.3)

$$v(x) = e^{i\xi x}w(x)$$

for some $w \in L^2_{\text{per}}(0,T)$ and $\xi \in [-\pi/T, \pi/T)$, where here we have suppressed the parameter dependence of $v$ and $w$. From these observations, it can be shown that $\lambda \in \mathbb{C}$ belongs to the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$ if and only if the problem

$$\begin{cases} \mathcal{A}[\phi]v = \lambda v \\ v(x+T) = e^{i\xi T}v(x) \end{cases} \tag{1.3.4}$$

admits a non-trivial solution for some $\xi \in [-\pi/T, \pi/T)$. Equivalently, substituting (1.3.3) into (1.3.4), we see that (1.3.4) holds if and only if there exists a $\xi \in [-\pi/T, \pi/T)$ and a non-trivial $w \in L^2_{\text{per}}(0,T)$ such that

$$\lambda w = e^{-i\xi x}\mathcal{A}[\phi]e^{i\xi x}w =: \mathcal{A}_\xi[\phi]w. \tag{1.3.5}$$

For details, see [40, 55, 28], for example. The one-parameter family of operators $\mathcal{A}_\xi[\phi]$ are called the *Bloch operators* associated to $\mathcal{A}[\phi]$, and $\xi$ is referred to as the *Bloch parameter* or sometimes as the *Bloch frequency*. Observe that $\mathcal{A}_0[\phi]$ corresponds to considering the operator $\mathcal{A}[\phi]$ with $T$-periodic, i.e., co-periodic, boundary conditions. Note that each $\mathcal{A}_\xi[\phi]$ acts on the space of

[10]In particular, such solutions can not have finite norm in $L^p(\mathbb{R})$ for any $1 \le p < \infty$.

$T$-periodic functions $L^2_{\text{per}}(0,T)$, on which they are closed with dense and compactly embedded domain $H^s_{\text{per}}(0,T)$.[11]

Since the domains of the Bloch operators are compactly embedded in $L^2_{\text{per}}(0,T)$, their spectra consist entirely of isolated eigenvalues of finite algebraic multiplicities[12] which, furthermore, depend continuously on the Bloch parameter $\xi$. By the above Floquet-Bloch theory, we in fact have the spectral decomposition

$$\sigma_{L^2(\mathbb{R})}\left(\mathcal{A}[\phi]\right) = \bigcup_{\xi\in[-\pi/T,\pi/T)} \sigma_{L^2_{\text{per}}(0,T)}\left(\mathcal{A}_\xi[\phi]\right); \tag{1.3.6}$$

see, for example, [17]. This characterizes the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$ as the union of countably many continuous curves $\lambda(\xi)$ corresponding to the eigenvalues of the associated Bloch operators $\mathcal{A}_\xi[\phi]$. For more details, see [55].

### 1.3.2 Notions of Spectral Stability

Equipped with this new understanding of the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$, we now introduce two more notions of spectral stability. In particular, when (1.1.1) is translationally invariant, the above characterization of the spectrum of $\mathcal{A}[\phi]$ acting on $L^2(\mathbb{R})$ implies that the spectrum necessarily touches the imaginary axis at the origin. Consequently, we introduce the following notion of stability that will be used throughout this dissertation.

**Definition 1.3.1** (Diffusive Spectral Stability)**.** Let $\phi \in L^2_{\text{per}}(0,T)$ be a stationary solution of (1.1.1). Then $\phi$ is said to be *diffusively spectrally stable* provided the following conditions hold:

(i) the spectrum of the linear operator $\mathcal{A}[\phi]$ acting on $L^2(\mathbb{R})$ satisfies

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}[\phi]) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\} \cup \{0\};$$

(ii) there exists $\theta > 0$ such that for any $\xi \in [-\pi/T, \pi/T)$ the real part of the spectrum of the Bloch

[11] In particular, note that the space where $\phi$ lives is now consistent with the space where $\mathcal{A}_\xi[\phi]$ acts.

[12] In other words, the $L^2_{\text{per}}(0,T)$-spectra of the Bloch operators is entirely point spectra.

operator $\mathcal{A}_\xi[\phi] := e^{-i\xi x}\mathcal{A}[\phi]e^{i\xi x}$ acting on $L^2_{\text{per}}(0,T)$ satisfies

$$\Re\left(\sigma_{L^2_{\text{per}}(0,T)}(\mathcal{A}_\xi[\phi])\right) \leq -\theta\xi^2;$$

(iii) $\lambda = 0$ is a simple $T$-periodic eigenvalue of $\mathcal{A}_0[\phi]$ with associated eigenfunction $\phi'$.[13]

We note that since the pioneering work of Schneider [58, 59, 60] the above has been the standard spectral stability assumption in nonlinear stability results for periodic traveling or standing waves in dissipative systems: see, for example, the more recent works [13, 29, 30, 31, 56] and references therein. In particular, this is the best one can hope for in the case of translationally invariant periodic patterns, and extensions of this to systems with more symmetries (hence more spectral curves passing through the origin) are regularly used: see, for example, [2, 31]. Most importantly, the above diffusive spectral stability assumption, and its extensions, has been shown to imply important details regarding the nonlinear dynamics near $\phi$ under localized, or general bounded, perturbations, including long-time dynamics of the associated modulation functions. For more information, see the above mentioned works.

To determine the (diffusive) spectral stability of a $T$-periodic wave $\phi$ when subjected to localized perturbations, one must therefore determine all of the $T$-periodic eigenvalues for each Bloch operator for $\xi \in [-\pi/T, \pi/T)$. In general, this is a very difficult task, and it may not be possible to rigorously understand the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$. However, the spectral decomposition (1.3.6), hence the (diffusive) spectral stability of $\phi$, can often be studied via well-conditioned numerical analysis.

While a rigorous understanding of the entire $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$ is not possible in general, that does not mean we cannot make any rigorous conclusions about the spectrum. Indeed, we usually know the $T$-periodic generalized kernel of $\mathcal{A}_0[\phi]$, i.e., $\text{gker}_{\text{per}}(\mathcal{A}_0[\phi])$, explicitly as a result of the symmetries in the PDE (1.1.1). Spectral perturbation theory then allows us to conclude spectral information about the $L^2_{\text{per}}(0,T)$-spectrum of $\mathcal{A}_\xi[\phi]$ for $|\xi| \ll 1$ near $\lambda = 0$. Hence, we often have a completely rigorous understanding of the spectrum of $\mathcal{A}[\phi]$ acting on $L^2(\mathbb{R})$ near the origin,

[13]In situations where (1.1.1) has more symmetries, this condition may be relaxed to incorporate the larger $T$-periodic generalized kernel of $\mathcal{A}_0[\phi]$.

$\lambda = 0$. This motivates the following notion of stability.

**Definition 1.3.2** (Spectral Modulational Stability)**.** Let $\phi \in L^2_{\mathrm{per}}(0,T)$ be a stationary solution of (1.1.1). Then $\phi$ is said to be *modulationally stable* if, in a sufficiently small neighborhood of the origin, the spectrum of $\mathcal{A}[\phi]$ acting on $L^2(\mathbb{R})$ lies in the closed left half-plane. In other words, $\phi$ is modulationally stable if, in a sufficiently small neighborhood of the origin, $\phi$ is spectrally stable. Otherwise, $\phi$ is said to be *modulationally unstable*.[14]

It is natural to ask what the physical relevance of Definition 1.3.2 is. Well, when studying periodic waves, there is a third important class of perturbations known as *long wavelength* perturbations. In particular, there is a special class of long wavelength perturbations that correspond to slow modulations of $\phi$. These slow modulations of $\phi$ correspond to perturbations that slowly vary, i.e., modulate, the fundamental characteristics of the periodic wave, such as amplitude or frequency, over large space-time scales. It has been shown in many recent works [3, 6, 38, 35, 32] that the spectral stability of $\phi$ when subjected to such slow modulations corresponds to the modulational stability described in Definition 1.3.2.

To motivate this, observe from (1.3.4) that the spectrum of $A_0[\phi]$ corresponds to the spectral stability of $\phi$ to $T$-periodic perturbations, i.e., to perturbations with the same period as the carrier wave. Similarly, (1.3.4) implies that each $|\xi| \ll 1$ corresponds to the spectral stability of $\phi$ to long wavelength perturbations of the carrier wave. As mentioned above, slow modulations of $\phi$ form a special class of long wavelength perturbations in which the effect of the perturbation is to slowly vary, namely modulate, the wave characteristics and the translational mode. These wave characteristics and the translational mode usually correspond to symmetries in the PDE (1.1.1). As such, variations in the wave characteristics and translational mode naturally provide spectral information about the co-periodic Bloch operator $\mathcal{A}_0[\phi]$ at the origin $\lambda = 0$. From the above considerations, it is natural to expect that the spectral stability of the underlying wave $\phi$ to slow modulations corresponds to the case when the $L^2_{\mathrm{per}}(0,T)$-spectrum of the Bloch operators $\mathcal{A}_\xi[\phi]$

[14]Note that modulational stability is a necessary condition for spectral stability in Definition 1.2.6, while modulational instability is a sufficient condition for spectral instability.

near $(\lambda, \xi) = (0,0)$ lie in the closed left half-plane, i.e., when $\phi$ is modulationally stable in the sense of Definition 1.3.2. For more discussion regarding this motivation, see [5].

### 1.3.3 Bloch Transform of Localized Functions

Now that we have an understanding of how to characterize the $L^2(\mathbb{R})$-spectrum of $\mathcal{A}[\phi]$, we seek to use this spectral information to understand the linear solution operator $e^{\mathcal{A}[\phi]t}$. In that direction, note that, from the earlier characterization of the spectrum of $\mathcal{A}[\phi]$, it is clearly desirable to have the ability to decompose arbitrary functions in $L^2(\mathbb{R})$ into superpositions of functions of the form $e^{i\xi x}w(x)$ with $\xi \in [-\pi/T, \pi/T)$ and $w \in L^2_{\text{per}}(0,T)$, see (1.3.3). This may be accomplished by noting that any function $g \in L^2(\mathbb{R})$ admits a Bloch decomposition, or inverse Bloch transform representation, given by

$$g(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}\check{g}(\xi, x)d\xi, \quad \text{where } \check{g}(\xi, x) := \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T}\widehat{g}(\xi + 2\pi\ell/T) \tag{1.3.7}$$

and where $\widehat{g}(\cdot)$ denotes the Fourier transform of $g$, defined here as

$$\widehat{g}(k) = \int_{-\infty}^{\infty} e^{-ikx}g(x)dx, \quad k \in \mathbb{R}.$$

Indeed, observe that for any Schwartz function $g$ we have that

$$2\pi g(x) = \int_{-\infty}^{\infty} e^{i\xi x}\widehat{g}(\xi)d\xi = \sum_{\ell\in\mathbb{Z}}\int_{-\pi/T}^{\pi/T} e^{i(\xi+2\pi\ell/T)x}\widehat{g}(\xi + 2\pi\ell/T)d\xi = \int_{-\pi/T}^{\pi/T} e^{i\xi x}\check{g}(\xi, x)d\xi,$$

and then the general result follows by density. Note that for each fixed $\xi \in [-\pi/T, \pi/T)$ the function $\check{g}(\xi, \cdot)$ is $T$-periodic and hence the above procedure decomposes, as desired, arbitrary functions in $L^2(\mathbb{R})$ into a (continuous) superposition of functions of the form $e^{i\xi\cdot}\check{g}(\xi, \cdot)$, each of which has a fixed Bloch frequency $\xi$.

Defining the Bloch transform

$$\mathcal{B} : L^2(\mathbb{R}) \to L^2\left([-\pi/T, \pi/T); L^2_{\text{per}}(0,T)\right)$$

via the action $g \mapsto \check{g}$ with $\check{g}$ given by (1.3.7), the standard Parseval identity for the Fourier transform implies that the Bloch transform is a bounded linear operator,

$$\|g\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\int_0^T |\mathcal{B}(g)(\xi,x)|^2\, dx\, d\xi = \frac{1}{2\pi T}\|\mathcal{B}(g)\|^2_{L^2\left([-\pi/T,\pi/T);L^2_{\text{per}}(0,T)\right)}. \tag{1.3.8}$$

Furthermore, for $v \in H^s(\mathbb{R})$ we have

$$\mathcal{B}\left(\mathcal{A}[\phi]v\right)(\xi,x) = \left(\mathcal{A}_\xi[\phi]\check{v}(\xi,\cdot)\right)(x)$$

and

$$\mathcal{A}[\phi]v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}\mathcal{A}_\xi[\phi]\check{v}(\xi,x)d\xi,$$

showing that the Bloch transform $\mathcal{B}$ diagonalizes the $T$-periodic coefficient linear operator $\mathcal{A}[\phi]$ in the same way that the standard Fourier transform diagonalizes constant coefficient linear operators. Here, the Bloch operators $\mathcal{A}_\xi[\phi]$ may be viewed as operator-valued symbols of the linearized operator $\mathcal{A}[\phi]$ under the action of $\mathcal{B}$. Furthermore, assuming that $\mathcal{A}[\phi]$ and its associated Bloch operators $\mathcal{A}_\xi[\phi]$ generate $C^0$-semigroups on $L^2(\mathbb{R})$ and $L^2_{\text{per}}(0,T)$, respectively, it is straightforward to check the identities

$$\mathcal{B}\left(e^{\mathcal{A}[\phi]t}v\right)(\xi,x) = \left(e^{\mathcal{A}_\xi[\phi]t}\check{v}(\xi,\cdot)\right)(x)$$

and

$$e^{\mathcal{A}[\phi]t}v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\check{v}(\xi,x)d\xi. \tag{1.3.9}$$

Combined with (1.3.8), this latter identity allows us to conclude information about the semigroup $e^{\mathcal{A}[\phi]t}$ acting on $L^2(\mathbb{R})$ by synthesizing, over $\xi \in [-\pi/T, \pi/T)$, information about the Bloch semigroups $e^{\mathcal{A}_\xi[\phi]t}$ acting on $L^2_{\text{per}}(0,T)$. As we will see in later chapters, this decomposition is key to understanding the linear stability of $\phi$ in the presence of localized perturbations.

### 1.3.4 Floquet-Bloch Theory for Subharmonic Perturbations

In this and the following section, we *introduce* a version of Floquet-Bloch theory that is appropriate for the study of subharmonic perturbations. Following the intuition outlined at the end of Section 1.1, we know that localized perturbations can formally be understood as the limit of subharmonic perturbations as $N \to \infty$. One of our main goals in later chapters is to exploit this formal intuition in our study of subharmonic perturbations. Consequently, we will use the Floquet-Bloch theory for localized perturbations outlined in the previous section as motivation for the development of a Floquet-Bloch theory that is appropriate for the study of subharmonic perturbations. Specifically, given a differential operator $\mathcal{A}[\phi]$ with $T$-periodic coefficients, as in the previous section, we aim at understanding both the $L^2_{\text{per}}(0, NT)$-spectrum of $\mathcal{A}[\phi]$ and how the associated semigroup $e^{\mathcal{A}[\phi]t}$ acts on $NT$-periodic functions.

To begin our study of the $L^2_{\text{per}}(0, NT)$-spectrum of $\mathcal{A}[\phi]$, recall from earlier in Section 1.3 that non-trivial solutions of the spectral problem (1.3.1)

$$\mathcal{A}[\phi]v = \lambda v$$

cannot be integrable on $\mathbb{R}$ and that, at best, they are bounded functions of the form (1.3.3)

$$v(x) = e^{i\xi x} w(x)$$

for some $w \in L^2_{\text{per}}(0, T)$ and $\xi \in [-\pi/T, \pi/T)$, where here we have suppressed the parameter dependence of $v$ and $w$. Recalling that the domain of $\mathcal{A}[\phi]$ is $H^s_{\text{per}}(0, NT)$ for some $s \geq 1$, we have that the domain of the linear operator is compactly embedded in $L^2_{\text{per}}(0, NT)$. Therefore, the spectrum of $\mathcal{A}[\phi]$ acting on $L^2_{\text{per}}(0, NT)$ consists entirely of isolated eigenvalues of finite algebraic multiplicities; hence, it is entirely point spectrum. Now observe that if we fix $N \in \mathbb{N}$ and define the set

$$\Omega_N := \left\{\xi \in [-\pi/T, \pi/T) \mid e^{i\xi NT} = 1\right\}, \tag{1.3.10}$$

then the perturbation $v$ satisfies $NT$-periodic boundary conditions if and only if $\xi \in \Omega_N$. In particular, it can be shown that $\lambda \in \mathbb{C}$ belongs to the $L^2_{\text{per}}(0, NT)$-spectrum of $\mathcal{A}[\phi]$ if and only if there exists a $\xi \in \Omega_N$ and a non-trivial $w \in L^2_{\text{per}}(0, T)$ such that

$$\mathcal{A}_\xi[\phi]w = \lambda w,$$

where the Bloch operator $\mathcal{A}_\xi[\phi]$ is defined as in (1.3.5). As before, each $\mathcal{A}_\xi[\phi]$ acts on $L^2_{\text{per}}(0, T)$ with densely defined and compactly embedded domain $H^s_{\text{per}}(0, T)$, and hence their spectra consists entirely of isolated eigenvalues with finite algebraic multiplicities which, furthermore, depend continuously on $\xi$. In fact, we have the spectral decomposition

$$\sigma_{L^2_{\text{per}}(0,NT)}\left(\mathcal{A}[\phi]\right) = \bigcup_{\xi \in \Omega_N} \sigma_{L^2_{\text{per}}(0,T)}\left(\mathcal{A}_\xi[\phi]\right). \tag{1.3.11}$$

This characterizes the $NT$-periodic spectrum of $\mathcal{A}[\phi]$ as the union of $T$-periodic eigenvalues for the Bloch operators $\{\mathcal{A}_\xi[\phi]\}_{\xi \in \Omega_N}$.

**Remark 1.3.3.** *In particular,* (1.3.11) *implies that diffusively spectrally stable periodic waves, in the sense of Definition 1.3.1, are spectrally stable for all subharmonic perturbations. Since the* $L^2_{\text{per}}(0, NT)$*-spectrum of* $\mathcal{A}[\phi]$ *is purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities, if* $\phi$ *is diffusively spectrally stable, then we must have* $\lambda = 0$ *is a simple eigenvalue – with associated eigenfunction the derivative* $\phi'$ *of the periodic wave – and that the remaining eigenvalues have negative real parts, satisfying the spectral gap condition*

$$\Re\left(\sigma_{L^2_{\text{per}}(0,NT)}(\mathcal{A}[\phi]) \setminus \{0\}\right) \leq -\delta_N,$$

*for some* $\delta_N > 0$*. Because the eigenvalues of the Bloch operators* $\mathcal{A}_\xi[\phi]$ *depend continuously on* $\xi$*,* *it is not difficult to see that the spectral gap $\delta_N$ above tends to* $0$*, as $N \to \infty$. For more information on the implications of this remark, see Chapters 3 and 4 below.*

**Remark 1.3.4.** *For definiteness, we note that the set $\Omega_N$ given by* (1.3.10) *may be written explicitly when $N$ is even by*

$$\Omega_N = \left\{\xi_j = \frac{2\pi j}{NT} : j = -\frac{N}{2},\ -\frac{N}{2}+1,\ldots,\frac{N}{2}-1\right\}$$

*and when $N$ is odd by*

$$\Omega_N = \left\{\xi_j = \frac{2\pi j}{NT} : j = -\frac{N-1}{2},\ -\frac{N-1}{2}+1,\ldots,\frac{N-1}{2}\right\}.$$

*In particular, observe that we have $0 \in \Omega_N$ and $|\Omega_N| = N$ for all $N \in \mathbb{N}$ and that, furthermore, $\Delta\xi_j := \xi_j - \xi_{j-1} = \frac{2\pi}{NT}$ for each appropriate $j$.*

### 1.3.5 Bloch Transform of Subharmonic Functions

We now seek to use the spectral decomposition (1.3.11) to understand how the linear solution operator $e^{\mathcal{A}[\phi]t}$ acts on $NT$-periodic functions. From the discussion above, it is clearly desirable to have the ability to decompose arbitrary functions in $L^2_{\text{per}}(0,NT)$ into superpositions of functions of the form $e^{i\xi x}w(x)$ with $\xi \in \Omega_N$ and $w \in L^2_{\text{per}}(0,T)$. This is achieved by noting that a given $g \in L^2_{\text{per}}(0,NT)$ admits a Fourier series representation

$$g(x) = \frac{1}{NT}\sum_{k\in\mathbb{Z}} e^{2\pi ikx/NT}\widehat{g}(2\pi k/NT),$$

where here $\widehat{g}$ represents the Fourier transform of $g$ on the torus given by

$$\widehat{g}(\zeta) = \int_{-NT/2}^{NT/2} e^{-i\zeta y} g(y)dy. \tag{1.3.12}$$

Note that $\widehat{g}(\zeta)$ is the $k$-th Fourier coefficient of the $NT$-periodic function $g$ when $\zeta = 2\pi k/NT$. Together with the identity (valid for any $f$ for which the sum converges)

$$\sum_{k\in\mathbb{Z}} f(2\pi k/NT) = \sum_{\xi\in\Omega_N}\sum_{\ell\in\mathbb{Z}} f(\xi + 2\pi\ell/T),$$

it follows that $g$ may be represented as

$$g(x) = \frac{1}{NT}\sum_{\xi\in\Omega_N}\sum_{\ell\in\mathbb{Z}} e^{i(\xi+2\pi\ell/T)x}\widehat{g}(\xi + 2\pi\ell/T).$$

In particular, defining for $\xi \in \Omega_N$ the $T$-periodic Bloch transform of a function $g \in L^2_{\text{per}}(0, NT)$ as

$$\mathcal{B}_T(g)(\xi, x) := \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T}\widehat{g}(\xi + 2\pi\ell/T), \tag{1.3.13}$$

the above yields the inverse Bloch representation formula

$$g(x) = \frac{1}{NT}\sum_{\xi\in\Omega_N} e^{i\xi x}\mathcal{B}_T(g)(\xi, x) \tag{1.3.14}$$

which is valid for all $g \in L^2_{\text{per}}(0, NT)$. Note that the function $\mathcal{B}_T(g)(\xi, \cdot)$ is clearly $T$-periodic for each $\xi \in \Omega_N$, and hence the above representation formula decomposes arbitrary $NT$-periodic functions in the desired fashion. Further note that the equalities (1.3.13) and (1.3.14) are the analogue of (1.3.7) for $NT$-periodic functions.

Before proceeding, we note that, in fact, the $T$-periodic Bloch transform

$$\mathcal{B}_T : L^2_{\text{per}}(0, NT) \to \ell^2\left(\Omega_N : L^2_{\text{per}}(0, T)\right)$$

as defined above satisfies the subharmonic Parseval identity

$$\langle f, g\rangle_{L^2(0,NT)} = \frac{1}{NT^2}\sum_{\xi\in\Omega_N} \langle\mathcal{B}_T(f)(\xi, \cdot), \mathcal{B}_T(g)(\xi, \cdot)\rangle_{L^2(0,T)}\,, \tag{1.3.15}$$

valid for all $f, g \in L^2_{\text{per}}(0, NT)$. Indeed, observe that

$$\begin{aligned}
\sum_{\xi \in \Omega_N} \langle \mathcal{B}_T(f)(\xi, \cdot), \mathcal{B}_T(g)(\xi, \cdot) \rangle_{L^2(0,T)} &= \sum_{\xi \in \Omega_N} \int_0^T \overline{\mathcal{B}_T(f)(\xi, x)} \mathcal{B}_T(g)(\xi, x) dx \\
&= \sum_{\xi \in \Omega_N} \int_0^T \sum_{k,\ell \in \mathbb{Z}} e^{2\pi i(\ell - k)x/T} \overline{\hat{f}(\xi + 2\pi k/T)} \hat{g}(\xi + 2\pi \ell/T) dx \\
&= \sum_{\xi \in \Omega_N} \sum_{k,\ell \in \mathbb{Z}} \left( \int_0^T e^{2\pi i(\ell - k)x/T} dx \right) \overline{\hat{f}(\xi + 2\pi k/T)} \hat{g}(\xi + 2\pi \ell/T) \\
&= T \sum_{\xi \in \Omega_N} \sum_{\ell \in \mathbb{Z}} \overline{\hat{f}(\xi + 2\pi \ell/T)} \hat{g}(\xi + 2\pi \ell/T) \\
&= T \sum_{k \in \mathbb{Z}} \overline{\hat{f}\left(\frac{2\pi k}{NT}\right)} \hat{g}\left(\frac{2\pi k}{NT}\right) = NT^2 \langle f, g \rangle_{L^2(0,NT)},
\end{aligned}$$

as claimed. In particular, this yields the useful identity

$$\|g\|^2_{L^2(0,NT)} = \frac{1}{NT^2} \sum_{\xi \in \Omega_N} \|\mathcal{B}_T(g)(\xi, \cdot)\|^2_{L^2(0,T)}$$

valid for all $g \in L^2_{\text{per}}(0, NT)$, establishing that (up to normalization) $\mathcal{B}_T$ is an isometry, just as the Bloch transform $\mathcal{B}$ in the case of localized perturbations. (1.3.15) also leads to the following lemma needed later in our analysis.

**Lemma 1.3.5.** *Let $N \in \mathbb{N}$. If $f \in L^2_{\text{per}}(0, T)$ and $g \in L^2_{\text{per}}(0, NT)$, then*

$$\mathcal{B}_T(fg)(\xi, x) = f(x)\mathcal{B}_T(g)(\xi, x),$$

*In particular, for such $f$ and $g$ we have the identity*

$$\langle f, g \rangle_{L^2(0,NT)} = \frac{1}{T} \langle f(x), \mathcal{B}_T(g)(0, x) \rangle_{L^2(0,T)}.$$

*Proof.* For $f \in L^2_{\mathrm{per}}(0,T)$ and $g \in L^2_{\mathrm{per}}(0,NT)$, we calculate the $NT$-periodic Fourier transform as

$$\begin{aligned}
\widehat{fg}(z) &= \int_0^{NT} e^{-izy} f(y)g(y)dy \\
&= \int_0^{NT} e^{-izy} \left( \frac{1}{T} \sum_{k\in\mathbb{Z}} e^{2\pi iky/T} \hat{f}\,(2\pi k/T) \right) g(y)dy \\
&= \frac{1}{T} \sum_{k\in\mathbb{Z}} \hat{f}\,(2\pi k/T) \left( \int_0^{NT} e^{-i(z-2\pi k/T)y} g(y)dy \right) \\
&= \frac{1}{T} \sum_{k\in\mathbb{Z}} \hat{f}\,(2\pi k/T)\, \hat{g}(z - 2\pi k/T).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathcal{B}_T(fg)(\xi,x) &= \frac{1}{T} \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T} \left( \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k/T)\hat{g}(\xi + 2\pi(\ell - k)/T) \right) \\
&= \frac{1}{T} \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k/T) \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T} \hat{g}\,(\xi + 2\pi(\ell - k)/T) \\
&= \frac{1}{T} \sum_{k\in\mathbb{Z}} e^{2\pi ikx/T} \hat{f}\,(2\pi k/T) \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T} \hat{g}\,(\xi + 2\pi\ell/T) \\
&= f(x)\mathcal{B}_T(g)(\xi,x),
\end{aligned}$$

which proves the first equality.

Next, from this equality and Parseval's identity (1.3.15) we obtain that

$$\langle f, g \rangle_{L^2(0,NT)} = \frac{1}{NT^2} \sum_{\xi\in\Omega_N} \langle f(x)\mathcal{B}_T(1)(\xi,\cdot), \mathcal{B}_T(g)(\xi,\cdot) \rangle_{L^2(0,T)}\,.$$

Noting that

$$\hat{1}\,(\xi + 2\pi\ell/T) = \begin{cases} NT, & \text{if } \ \xi + 2\pi\ell/T = 0 \\ 0, & \text{otherwise} \end{cases}$$

and that the condition $\xi + 2\pi\ell/T = 0$ holds for $\xi \in \Omega_N$ and $\ell \in \mathbb{Z}$ if and only if $\xi = \ell = 0$, we find that $\mathcal{B}_T(1)(\xi,x) = NT$, which proves the second equality. □

Similarly to the case for localized perturbations, we note that for $v \in H^s_{\text{per}}(0, NT)$

$$\mathcal{B}_T\left(\mathcal{A}[\phi]v\right)(\xi, x) = \left(\mathcal{A}_\xi[\phi]\mathcal{B}_T(v)(\xi, \cdot)\right)(x)$$

and

$$\mathcal{A}[\phi]v(x) = \frac{1}{NT}\sum_{\xi\in\Omega_N} e^{i\xi x}\mathcal{A}_\xi[\phi]\mathcal{B}_T(v)(\xi, x).$$

Hence we may view the Bloch operators $\mathcal{A}_\xi[\phi]$ as operator valued symbols associated to $\mathcal{A}[\phi]$ under the action of the $T$-periodic Bloch transform $\mathcal{B}_T$. If the operator $\mathcal{A}[\phi]$ and its corresponding Bloch operators $\mathcal{A}_\xi[\phi]$ generate $C^0$-semigroups on $L^2_{\text{per}}(0, NT)$ and $L^2_{\text{per}}(0, T)$, respectively, it is now straightforward to check the identities

$$\mathcal{B}_T\left(e^{\mathcal{A}[\phi]t}v\right)(\xi, x) = \left(e^{\mathcal{A}_\xi[\phi]t}\mathcal{B}_T(v)(\xi, \cdot)\right)(x)$$

and

$$e^{\mathcal{A}[\phi]t}v(x) = \frac{1}{NT}\sum_{\xi\in\Omega_N} e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\mathcal{B}_T(v)(\xi, x). \tag{1.3.16}$$

Combined with (1.3.15), this latter identity allows us to conclude information about the semigroup $e^{\mathcal{A}[\phi]t}$ acting on $L^2_{\text{per}}(0, NT)$ by synthesizing (over $\xi \in \Omega_N$) information about the Bloch semigroups $e^{\mathcal{A}_\xi[\phi]t}$ acting on $L^2_{\text{per}}(0, T)$. As we will see in later chapters, this decomposition is key to understanding the linear stability of $\phi$ in the presence of subharmonic perturbations.

An important observation here is that since (1.3.14) can be rewritten as

$$g(x) = \frac{1}{2\pi}\sum_{\xi\in\Omega_N} e^{i\xi x}\mathcal{B}_T(g)(\xi, x)\Delta\xi, \quad \Delta\xi = \frac{2\pi}{NT},$$

the representation (1.3.14) has the form of a Riemann sum approximation of the Bloch decomposition formula (1.3.7) for functions $g \in L^2(\mathbb{R})$. Similarly, the representation in (1.3.16) may be considered as a Riemann sum approximation of the formula (1.3.9). These interpretations of the above identities will be crucial in the forthcoming chapters.[15]

**Remark 1.3.6.** *In the case $N = 1$, corresponding to co-periodic perturbations, we have $\Omega_1 = \{0\}$ and hence the Bloch transform $\mathcal{B}_T(g)$ simply recovers the Fourier series representation for $g \in L^2_{\text{per}}(0, T)$. Furthermore, the representation formulas for $\mathcal{A}[\phi]$ and $e^{\mathcal{A}[\phi]t}$ given above are reduced to the obvious equalities $\mathcal{A}[\phi] = \mathcal{A}_0[\phi]$ and $e^{\mathcal{A}[\phi]t} = e^{\mathcal{A}_0[\phi]t}$.*

## 1.4 Outline of Dissertation

The outline for the rest of the dissertation is as follows. In Chapter 2 we study the modulational stability of $T$-periodic traveling waves in the context of viscous fluid conduits described by the conduit equation. Formally, the behavior of such modulated periodic waves on large space and time scales is well described by Whitham's theory of wave modulations, i.e., Whitham theory. Our goal for this chapter is to rigorously justify the connection between Whitham theory and modulational stability for the conduit equation following some of the intuition provided in Section 1.3.2 above.

In Chapter 3 we study the linear stability of $T$-periodic standing waves in the context of the Lugiato-Lefever equation (LLE)[16] when subjected to both localized and subharmonic perturbations. One of our main goals for this chapter is to establish a linear stability result to subharmonic perturbations that is *uniform* in $N$ in order to overcome a degeneracy that happens in the limit as $N \to \infty$. Throughout this first chapter, we have motivated the idea that the study of localized perturbations provide the necessary intuition to establish a uniform in $N$ result. We make this idea more concrete in Chapter 3 through the introduction of a new methodology.

We will see in the context of the LLE that, when trying to upgrade our linear results to the nonlinear level, the standard nonlinear iteration scheme breaks down due to an unavoidable "loss of regularity." However, we know that in cases where the dissipation is sufficiently strong, we can use a "nonlinear damping" result to regain this lost regularity, at least in the case of localized perturbations. Consequently, this motivates the structure for the rest of the dissertation. Recalling

[15]Further notice that the subharmonic Parseval identity (1.3.15) can be interpreted as a Riemann sum approximation of the localized Parseval identity (1.3.8), though this interpretation is less important for our analysis.

[16]The LLE is a damped nonlinear Schödinger equation with forcing that arises is nonlinear optics.

that the localized case served as motivation for the uniform study of subharmonic perturbations, we first seek to understand if it is possible – in the presence of sufficiently strong damping, allowing one to circumvent the loss of regularity via a nonlinear damping result – to establish a nonlinear stability result to subharmonic perturbations that is uniform in the period of the perturbations. This is the focus of Chapter 4, where we study the uniform nonlinear stability of $T$-periodic travelling waves when subjected to subharmonic perturbations in the context of systems of reaction-diffusion equations.

Chapter 4 confirms that methodologies used to circumvent the loss of regularity that occurs in our nonlinear iteration schemes for localized perturbations can motivate an appropriate methodology that can similarly circumvent the loss of regularity in the case of subharmonic perturbations. Consequently, in Chapter 5, we return to the context of the Lugiato-Lefever equation in order to develop a new methodology that allows us to circumvent the loss of regularity in the case of localized perturbations and in the presence of weak damping, i.e., in the absence of a nonlinear damping estimate. With this methodology in hand, along with the intuition provided from Chapter 4, we are currently studying the ability to use this to establish the desired uniform subharmonic stability result for the Lugiato-Lefever equation [23].

# Chapter 2

# Spectral Modulational Instability of Periodic Waves in Viscous Fluid Conduits

## Abstract

In this chapter, we are interested in studying the modulational dynamics of interfacial waves rising buoyantly along a conduit of a viscous liquid. Formally, the behavior of modulated periodic waves on large space and time scales may be described through the use of Whitham modulation theory. The application of Whitham theory, however, is based on formal asymptotic (WKB) methods, thus removing a layer of rigor that would otherwise support their predictions. In this study, we aim at rigorously verifying the predictions of the Whitham theory, as it pertains to the modulational stability of periodic waves, in the context of the so-called conduit equation, a nonlinear dispersive PDE governing the evolution of the circular interface separating a light, viscous fluid rising buoyantly through a heavy, more viscous, miscible fluid at small Reynolds numbers. In particular, using rigorous spectral perturbation theory, we connect the predictions of Whitham theory to the rigorous spectral (in particular, modulational) stability of the underlying wave trains. This makes rigorous recent formal results on the conduit equation obtained by Maiden and Hoefer. This chapter is representative of [32], which was a joint work with Mathew Johnson.

## 2.1 Introduction

In this chapter, we consider the modulation of periodic traveling wave solutions to the conduit equation

$$u_t + (u^2)_x - (u^2(u^{-1}u_t)_x)_x = 0, \tag{2.1.1}$$

which was derived in [53] to model the evolution of a circular interface separating a light, viscous fluid rising buoyantly through a heavy, more viscous, miscible fluid at small Reynolds numbers. In (2.1.1), $u = u(x,t)$ denotes a nondimensional cross-sectional area of the interface at nondimensional vertical coordinate $x$ and nondimensional time $t$: see Figure 2.1(a). The conduit equation (2.1.1) has also been studied in the geological context, where it is known to describe, under appropriate assumptions, the vertical transport of molten rock up a viscously deformable pipe (for example, narrow conduits and dykes) in the earth's crust. In that context, (2.1.1) is a special case of the more general "magma" equations [61, 62]

$$u_t + (u^n)_x - (u^n(u^{-m}u_t)_x)_x = 0 : \tag{2.1.2}$$

where here the parameters $n$ and $m$ correspond to permeability of the rock and the bulk viscosity, respectively. The physical regime for these exponents is $2 \leq n \leq 5$ and $0 \leq m \leq 1$: see [61]. Clearly, the conduit equation corresponds to (2.1.2) with $(n,m) = (2,1)$.

In contrast to magma, however, viscous fluid conduits are easily accessible in a laboratory setting: see, for example, [48] and references therein. Consequently, there has been quite a bit of study recently into the dynamics of solutions of the conduit equation (2.1.1) and their comparison to laboratory experiments. As described in [47], early experimental studies of viscous fluid conduits concentrated primarily on the formation of the conduit itself via the continuous injection of an intrusive viscous fluid into an exterior, miscible, much more viscous fluid, see [53]. Since then, a considerable amount of effort has been spent studying the dynamics and stability of solitary waves, as well as soliton-soliton interactions [53, 26, 65, 44]. More recently, it has been observed

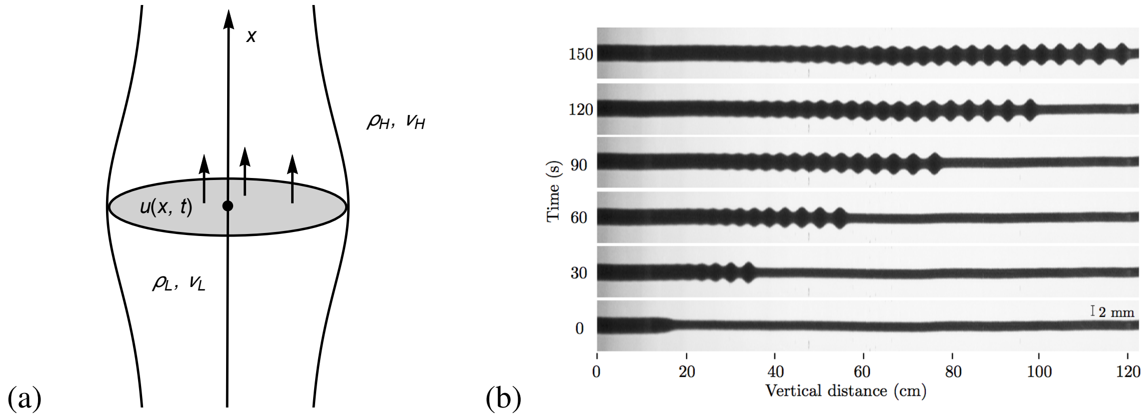



Figure 2.1: (a) A schematic drawing for the conduit equation. In the physical system, denoting the densities and viscosities of the heavy (outer) and light (inner) fluids as $\rho_H, \nu_H$ and $\rho_L, \nu_L$, respectively, the conduit equation holds under the assumption that $\rho_H > \rho_L$ and $\nu_H \gg \nu_L$. The arrows represent rising due to buoyancy. (b) The formation of periodic wave trains propagating in a physical experiment. Reprinted figure with permission from [48] Copyright (2019) by the American Physical Society.

experimentally that the competition of dispersive effects due to buoyancy and the nonlinear self-steepening effects of the surrounding media may result in the formation of dispersive shock waves (DSWs): see, for example, [48]. As described there, by adjusting the injection rate of the intrusive viscous fluid appropriately it was found that interfacial wave oscillations form behind a sharp, soliton-like leading edge, with the wider regions moving faster than narrower regions: see Figure 2.1(b). Such patterns correspond to dispersively regularized shock waves and have been the subject of much recent study due to their experimental realization [62, 1, 68]. Consequently, spatially modulated oscillations seem to form a fundamental building block regarding the long-time dynamics of the physical experiment. It is thus reasonable to expect that any reasonable mathematical model describing these physical experiments should admit oscillatory wave forms that are persistent (i.e., stable) when subject to slow wave modulations. Motivated by these observations, in this chapter we aim at studying the rigorous modulational, i.e., side-band, stability of periodic traveling wave forms in the conduit equation (2.1.1).

While the stability and dynamics of solitary waves of the magma and conduit equations has been studied extensively, as described above, a rigorous analysis of the local dynamics of periodic traveling waves seems lacking. Note this problem is complicated by the fact that while these equations are dispersive, they generally lack a Hamiltonian structure[1]. Most existing analyses seem to appeal to Whitham's theory of wave modulations: see, for example, [14, 43, 47, 50]. This theory proceeds by rewriting the governing PDE in slow coordinates $(X, S) = (\varepsilon x, \varepsilon t)$ then uses a multiple scale (WKB) approximation of the solution and seeks a homogenized system, known as the Whitham modulation equations, describing the mean behavior of the resulting approximation. This approach is widely used to describe the behavior of modulated periodic waves on large space and time scales, and in particular, is expected to predict the stability of periodic wavetrains to slow modulations (compare to Definition 1.3.2 and the discussion following). Specifically, hyperbolicity (i.e., local well-posedness) of the Whitham system about a periodic solution $\phi$ of the governing PDE is expected to be a necessary condition for the stability to slow modulations of $\phi$: see, for example, [70].

Whitham modulation theory has recently been applied to the conduit equation (2.1.1), where the authors, by coupling their analysis to numerical time evolution studies and numerical analysis of the Whitham system, identify an amplitude-dependent region of parameter space where such periodic wave trains are expected to be stable to slow modulations. However, we note the formal asymptotic methods used in Whitham theory are not, in general, rigorously justified, so the primary goal of this chapter is to (rigorously) connect the predictions from Whitham modulation theory to the rigorous dynamical stability of the underlying periodic wave train solutions of the conduit equation (2.1.1). Specifically, our main result, Theorem 2.3.1, establishes that hyperbolicity of the Whitham modulation equations about a periodic wave $\phi$ of (2.1.1) is indeed a necessary condition for the stability of $\phi$ to slow modulations. This will be accomplished in Section 2.4 by performing a rigorous analysis of the spectrum of the linearization of (2.1.1) about such periodic traveling waves $\phi$. Specifically, using Floquet-Bloch theory and spectral perturbation theory we show that the spectrum near the origin of the linearization of (2.1.1) about $\phi$ consists of three $C^1$ curves

[1]We note the magma equations (2.1.2) admit a Hamiltonian formulation only when $n + m = 0$, which is outside the parameter regime relevant to magma dynamics. See [66] for a nonlinear stability analysis in this seemingly nonphysical case.

which, locally, satisfy

$$\lambda_j(\xi) = i\alpha_j\xi + o(\xi)$$

where the $\alpha_j$ are precisely the characteristic speeds associated with the Whitham modulation equations about $\phi$. Consequently, a necessary condition for spectral stability of $\phi$ is that all the $\alpha_j$ are real, which is equivalent to the associated Whitham system being weakly hyperbolic at $\phi$. Such a rigorous connection between the stability of periodic waves and the Whitham modulation equations has been established previously in a number of contexts: see, for example, [6, 38, 5, 3, 63] and references therein. The specific approach here follows more closely the analysis in [3], which is based off the work of Serre in [63].

The organization of the chapter is as follows. In Section 2.2, we begin by recalling some basic facts about the conduit equation (2.1.1). Specifically, we discuss the conservation laws associated to (2.1.1) as well as the existence analysis for periodic traveling wave solutions. In Section 2.3, we derive the Whitham modulation equations associated with (2.1.1) and state our main result Theorem 2.3.1. We begin the proof of Theorem 2.3.1 in Section 2.4, where we perform a rigorous spectral stability calculation using spectral perturbation theory. Specifically, there we derive a $3 \times 3$ matrix which rigorously encodes the spectrum of the linearized operator associated with (2.1.1) about a periodic traveling wave near the origin in the spectral plane. The proof of our main result, providing a rigorous connection between the Whitham modulation equations and the rigorous spectral analysis in Section 2.4, is then given in Section 2.5. Finally, we end by analyzing our results for waves with asymptotically small oscillations in Section 2.6.

## 2.2 Basic Properties of the Conduit Equation

In this section, we collect some important basic facts about the conduit equation (2.1.1).

### 2.2.1 Conservation Laws & Conserved Quantities

First, we note that it is shown in [64, Corollary 5.7] that the conduit equation is globally well-posed for initial data $u(x,0) - 1 \in H^1(\mathbb{R})$, so long as $u(x,0)$ satisfies the physically reasonable requirement of being bounded away from zero. Further, even though the conduit equation admits nearly elastic solitonic collisions, it is shown through the failure of the Painlevé test to not be completely integrable [25]. Nevertheless, (2.1.1) admits (at least) the following two conservation laws:

$$\begin{cases} u_t + (u^2 - u^2(u^{-1}u_t)_x)_x = 0 \\ (u^{-1} + u^{-2}(u_x)^2)_t + (u^{-1}u_{xt} - u^{-2}u_x u_t - 2\ln|u|)_x = 0 \end{cases} \tag{2.2.1}$$

Notice (2.2.1)(i) is simply a restatement of (2.1.1), showing that the conduit equation itself corresponds to conservation of mass. The existence of more conservation laws for the general magma equations (2.1.2) was studied by Harris in [24]. There it was shown that (2.1.2) generally only admits two conservation laws[2]. However, this analysis was shown to be inconclusive in a couple of cases, one of which occurs when $m = 1$, $n \neq 0$, which the conduit equation (2.1.1) clearly falls into[3]. Consequently, it seems to be currently unknown if (2.1.1) admits more conservation laws or not, though Harris seems to think the existence of additional conservation laws unlikely.

Restricting to solutions that are $T$-periodic in the spatial variable $x$, the conservation laws (2.2.1) give rise to the following two conserved quantities:

$$M(u) := \int_0^T u\,dx, \quad Q(u) := \int_0^T \frac{u + (u_x)^2}{u^2}\,dx. \tag{2.2.2}$$

As we will see, these conserved quantities will play an important part in our forthcoming analysis. Note that the conserved quantity $M$ corresponds to conservation of mass, while the conservation of $Q$ does not seem to have a clear physical meaning [43].

[2]One conservation law is always the magma equation (2.1.2) itself, while the structure for the other law varies depending on the parameters $(n, m)$.

[3]The other case is given by $m = n + 1$, $n \neq 0$, where a third conservation is shown to exist.

### 2.2.2 Existence of Periodic Traveling Waves

Traveling wave solutions of (2.1.1) correspond to solutions of the form $u(x,t) = \phi(x-ct)$ for some wave profile $\phi(\cdot)$ and wave speed $c > 0$. The profile $\phi(z)$ is readily seen to be a stationary solution of the evolutionary equation

$$u_t - cu_z + (u^2)_z - (u^2(u^{-1}(u_t - cu_z))_z)_z = 0, \tag{2.2.3}$$

written here in the traveling coordinate $z = x - ct$. After a single integration, stationary solutions of (2.2.3) are seen to satisfy the second-order ODE

$$-2cE = -c\phi + \phi^2 + c\phi^2(\phi^{-1}\phi')', \tag{2.2.4}$$

where here $'$ denotes differentiation with respect to $z$ and $E \in \mathbb{R}$ is a constant of integration. Multiplying (2.2.4) by $\phi^{-3}\phi'$, the profile equation can be rewritten[4] as

$$-2cE\phi^{-3}\phi' = -c\phi^{-2}\phi' + \phi^{-1}\phi' + \tfrac{c}{2}\left[\phi^{-2}(\phi')^2\right]'$$

and hence may be reduced by quadrature to

$$\tfrac{1}{2}(\phi')^2 = E - \left(\tfrac{1}{c}\phi^2 \ln|\phi| + a\phi^2 + \phi\right), \tag{2.2.5}$$

where here $a \in \mathbb{R}$ is a second constant of integration. By standard phase plane analysis, the existence and non-existence of bounded solutions of (2.2.4) is determined entirely by the effective potential

$$V(\phi; a, c) := \tfrac{1}{c}\phi^2 \ln|\phi| + a\phi^2 + \phi.$$

Indeed, for a given $a, c \in \mathbb{R}$, a necessary and sufficient condition for the existence of periodic solutions of (2.2.5) is that $V(\cdot; a, c)$ has local minima. Furthermore, since (in the physical modeling)

[4]Alternatively, one can use the identity $2(\phi^{-1}\phi')' = \phi(\phi')^{-1}\left(\phi^{-2}(\phi')^2\right)'$.

the dependent variable $\phi$ represents the cross-sectional area of the viscous fluid conduit, we additionally require that the local minima occur for $\phi > 0$.

To characterize the parameters $(a, c)$ for which $V(\cdot; a, c)$ has a strictly positive local minima, we study the critical points of $V(\cdot; a, c)$. Noting that

$$V_\phi(\phi; a, c) = \tfrac{2}{c}\phi \ln|\phi| + \tfrac{1}{c}\phi + 2a\phi + 1, \quad V_{\phi\phi}(\phi; a, c) = \tfrac{2}{c}\left(\ln|\phi| + \tfrac{3}{2} + ac\right)$$

and

$$V_{\phi\phi\phi}(\phi; a, c) = \frac{2}{c\phi},$$

we see that, since $c > 0$, the derivative $V_\phi(\cdot; a, c)$ has a local maximum at $\phi_- := -e^{-(ac+3/2)}$ and a local minimum at $\phi_+ := e^{-(ac+3/2)}$. Since $V_\phi(\cdot; a, c)$ additionally satisfies

$$\lim_{\phi \to 0^+} V_\phi(\phi; a, c) = 1, \quad V_\phi(\phi_+; a, c) = 1 - \tfrac{2}{c}e^{-(ac+3/2)} < 1, \quad \text{and} \quad \lim_{\phi \to +\infty} V_\phi(\phi; a, c) = +\infty$$

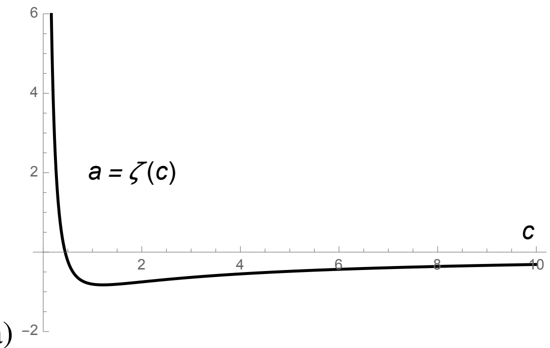
for all $a \in \mathbb{R}$, $c > 0$, it follows that the number of positive roots of $V_\phi(\cdot; a, c)$ for $c > 0$ is determined by the sign of the quantity $a - \zeta(c)$, where here

$$\zeta(c) := \frac{1}{c}\ln\left(\frac{2}{c}\right) - \frac{3}{2c}.$$

See Figure 2.2(a) for a plot of $a = \zeta(c)$. Indeed, if $a > \zeta(c)$, then $V_\phi(\phi_+; a, c)$ is positive and hence $V_\phi(\cdot; a, c)$ has no positive roots, while $a < \zeta(c)$ implies $V_\phi(\phi_+; a, c)$ is negative and hence $V_\phi(\cdot; a, c)$ has exactly two positive roots $0 < \phi_1 < \phi_2$. In the latter case, it is clear from the above analysis that $\phi_1$ and $\phi_2$ are local maxima and minima, respectively, of the effective potential[5] $V(\cdot; a, c)$: see Figure 2.2(b).

**Remark 2.2.1.** *We collect here some easily verifiable properties of the function $\zeta(c)$. First, $\zeta(c)$ only has one root, which occurs at $c = 2e^{-\frac{3}{2}}$, and one critical point (an absolute minimum), which*

[5]In the border case $a = \zeta(c)$, it follows that $V_\phi(\phi_+; a, c) = 0$. Using the above analysis, we may then conclude $\phi_+$ is a saddle point of the effective potential $V(\cdot; a, c)$.



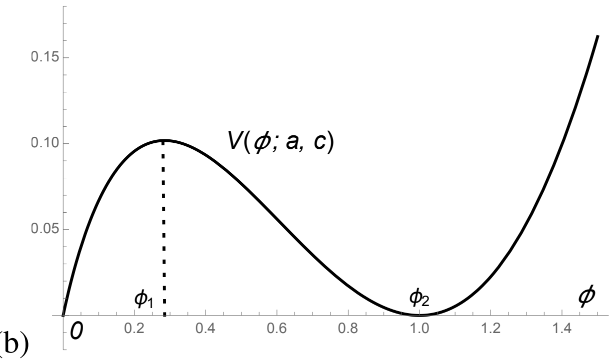




Figure 2.2: (a) A plot of $\zeta(c)$ for $c > 0$. Strictly positive periodic traveling wave solutions of (2.2.4) exist for $a < \zeta(c)$. (b) A plot of the effective potential $V(\phi; a, c)$ for $c = 1$ and $a = -1 < \zeta(1)$.

*occurs at* $c = 2e^{-1/2}$*. Furthermore,* $\lim_{c\to 0^+} \zeta(c) = +\infty$ *and* $\lim_{c\to+\infty} \zeta(c) = 0$*. See Figure 2.2(a) for a numerical plot.*

Returning to (2.2.5), it follows that if we define the set

$$\mathcal{B} := \left\{(a, E, c) \in \mathbb{R}^3 : c > 0, \;\; a < \zeta(c), \;\; E \in (V(\phi_2(a,c); a, c), V(\phi_1(a,c); a, c))\right\},$$

then for each $(a, E, c) \in \mathcal{B}$ the profile equation (2.2.4) admits a one-parameter family, parameterized by translation invariance, of strictly positive periodic solutions $\phi(x; a, E, c)$ with period

$$T = T(a, E, c) = \sqrt{2} \int_{\phi_{\min}}^{\phi_{\max}} \frac{d\phi}{\sqrt{E - V(\phi; a, c)}} = \frac{\sqrt{2}}{2} \oint_\gamma \frac{d\phi}{\sqrt{E - V(\phi; a, c)}},$$

where here $\phi_{\min} \in (\phi_1, \phi_2)$ and $\phi_{\max} \in (\phi_2, \infty)$ are roots of $E - V(\cdot; a, c)$ corresponding to the minimum and maximum values of the profile $\phi$, respectively, and integration over $\gamma$ represents a complete integration from $\phi_{\min}$ to $\phi_{\max}$, and then back to $\phi_{\min}$ again. Naturally, one must appropriately choose the branch of the square root in each direction. Alternatively, one could interpret the contour $\gamma$ as a closed (Jordan) curve in the complex plane $\mathbb{C}$ that encloses a bounded set containing both $\phi_{\min}$ and $\phi_{\max}$. Since the values $\phi_{\min/\max}$ are smooth functions of the traveling wave parameters $(a, E, c)$, a standard procedure shows the above integrals may be regularized at the square root branch points and hence represent $C^1$ functions of $a$, $E$, and $c$. In this way, we have proven the existence of a 4-parameter family (in fact, a $C^1$ manifold) of periodic traveling wave solutions of (2.2.4):

$$\phi(x - ct + x_0; a, E, c), \;\; x_0 \in \mathbb{R}, \;\; (a, E, c) \in \mathcal{B}$$

with period $T = T(a, E, c)$. Notice as $E \searrow V(\phi_2; a, c)$ the profile $\phi(\cdot; a, E, c)$ converges to a constant solution, while $T(a, E, c) \to +\infty$ as $E \nearrow V(\phi_1, a, c)$ which corresponds to a solitary wave limit. Without loss of generality, we may choose $x_0$ such that $\phi$ is an even function, which we will do throughout the rest of the chapter.

**Remark 2.2.2.** *In [65], the authors study the existence and nonlinear stability of traveling solitary waves of the more general class of magma equations* (2.1.2)*. In the context of the conduit equation* (2.1.1)*, the authors considered solitary waves satisfying $\phi \to 1$ as $|x| \to \infty$, which they show to exist for all $c > 2$. Taking $|x| \to \infty$ in* (2.2.4) *and requiring that $\phi = 1$ be a local maximum of $V(\cdot; a, c)$, we see these waves correspond to the choice $2cE = c - 1$, $a = E - 1$, and $c > 2$. See also Remark 2.4.4 below.*

By a similar procedure as above, the conserved quantities $M$ and $Q$ defined in (2.2.2) can be restricted to the manifold of periodic traveling wave solutions of (2.1.1). Indeed, given a $T$-periodic traveling wave $\phi(\cdot; a, E, c)$ of (2.2.3), we can define the functions $M, Q : \mathcal{B} \to \mathbb{R}$ via

$$\begin{cases} M(a, E, c) := \displaystyle\int_0^{T(a,E,c)} \phi(z; a, E, c)\, dz = \frac{\sqrt{2}}{2} \oint_\gamma \frac{\phi d\phi}{\sqrt{E - V(\phi; a, c)}} \\ Q(a, E, c) := \displaystyle\int_0^{T(a,E,c)} \frac{\phi(z; a, E, c) + (\phi'(z; a, E, c))^2}{(\phi(z; a, E, c))^2}\, dz = \frac{\sqrt{2}}{2} \oint_\gamma \frac{(\phi + 2(E - V(\phi; a, c)))\, d\phi}{\phi^2 \sqrt{E - V(\phi; a, c)}} \end{cases} \tag{2.2.6}$$

where the contour integral over $\gamma$ is defined as before. Following previous arguments, the above integrals can be regularized near their square root singularities and hence represent $C^1$ functions on $\mathcal{B}$. As we will see, the gradients of these conserved quantities along the manifold of periodic traveling wave solutions of (2.1.1) will play an important role in our analysis.

## 2.3 The Whitham Modulation Equations

In this section, we begin our study of the long-time dynamics of an arbitrary amplitude, slowly modulated periodic traveling wave solution of the conduit equation (2.1.1). An often used, yet completely formal, approach to study the dynamics of such slowly modulated periodic waves is to analyze the associated Whitham modulation equations [70]. While Whitham originally formulated this approach in terms of averaged conservation laws [69], it was later shown to be equivalent to an asymptotic reduction derived through formal multiple-scales (WKB) expansions [46]. For completeness, we recall the derivation in the context of the conduit equation (2.1.1): see also the description in [47, Appendix C].

To provide an asymptotic description of the slow modulation of periodic traveling wave solutions of (2.1.1), we separate both space and time into separate fast and slow scales. For $\varepsilon > 0$ sufficiently small, we introduce the "slow" variables $(X, S) := (\varepsilon x, \varepsilon t)$ and note that, in the slow coordinates, (2.1.1) can be written as

$$\varepsilon u_S + 2\varepsilon u u_X - \varepsilon^3 u u_{XXS} + \varepsilon^3 u_{XX} u_S = 0 \tag{2.3.1}$$

Following Whitham [69, 70], we seek solutions of (2.3.1) of the form

$$u(X, S; \varepsilon) = u^0\left(X, S, \tfrac{1}{\varepsilon}\psi(X, S)\right) + \varepsilon u^1\left(X, S, \tfrac{1}{\varepsilon}\psi(X, S)\right) + O(\varepsilon^2)$$

where here the phase $\psi$ is chosen to ensure that the functions $u^j$ are 1-periodic functions of the third coordinate $\theta = \psi(X, S)/\varepsilon$. Substituting this ansatz into (2.3.1) yields a hierarchy of equations in algebraic orders of $\varepsilon$ that must all be simultaneously satisfied. At the lowest order in $\varepsilon$, which here corresponds to $O(1)$, we find the relation

$$\psi_S u^0_\theta + \psi_X((u^0)^2)_\theta - \psi_X^2 \psi_S((u^0)^2((u^0)^{-1}u^0_\theta)_\theta)_\theta = 0. \tag{2.3.2}$$

After the identification $k := \psi_X$ and $\omega := -\psi_S$ as the spatial and temporal frequencies of the modulation, respectively, and $c := \frac{\omega}{k}$ as the wave speed, (2.3.2) is recognized, up to a global factor of $-k$, as the derivative with respect to the "fast" variable $\theta$ of the nonlinear profile equation (2.2.4), rescaled for 1-periodic functions. Note that, here, $k$, $\omega$ and $c$ are now functions of the slow variables $X$ and $S$. Consequently, for a fixed $X$ and $S$ we may choose $u^0$ to be a periodic traveling wave solution of (2.1.1), and hence of the form

$$u^0(\theta, X, S) = \phi(\theta, a(X,S), E(X,S), c(X,S))$$

for some even solution $\phi$ of (2.2.4) with $(a(X,S), E(X,S), c(X,S)) \in \mathcal{B}$. Notice the consistency condition $(\psi_X)_S = (\psi_S)_X$ implies the local wave number $k$ and wave speed $c$ must slowly evolve according to the relation

$$k_S + (kc)_X = 0, \tag{2.3.3}$$

which is sometimes referred to as "conservation of waves". Note that (2.3.3) effectively serves as a nonlinear dispersion relation. Indeed, in the case of linear waves one would have $\psi(X,S) = k(X - cS)$, which clearly satisfies (2.3.3).

Continuing to study the above hierarchy of equations, at $O(\varepsilon)$ we find

$$k\partial_\theta \mathcal{L}[u^0]u^1 = G[u^0]u^0_S + F(u^0), \tag{2.3.4}$$

where here $\mathcal{L}[u^0]$ and $G[u^0]$ are linear differential operators defined via

$$\begin{cases} \mathcal{L}[u^0] := \left(c - 2u^0 - k^2 c u^0_{\theta\theta}\right) + 2k^2 c u^0_\theta \partial_\theta - k^2 c u^0 \partial_\theta^2, \\ G[u^0] := \left(1 + k^2 u^0_{\theta\theta}\right) - k^2 u^0 \partial_\theta^2 \end{cases}$$

supplemented with 1-periodic boundary conditions, and

$$\begin{aligned} F(u^0) = ((u^0)^2)_X + \frac{3}{2}(k^2)_X c u^0 u^0_{\theta\theta} - \frac{1}{2}(k^2)_X c (u^0_\theta)^2 \\ + 2k^2 c_X u^0 u^0_{\theta\theta} + 2k^2 c u^0 u^0_{X\theta\theta} - k^2 c((u^0_\theta)^2)_X \end{aligned}$$

contains all the nonlinear terms in $u^0$ and its derivatives. Treating (2.3.4) as a forced linear equation for the unknown $u^1$, it follows by the Fredholm alternative that (2.3.4) is solvable in the class of 1-periodic functions if and only if

$$G[u^0]u_S^0 + F(u^0) \perp \ker_{\mathrm{L}^2_{\mathrm{per}}(0,1)} \left(\mathcal{L}^\dagger[u^0]\partial_\theta\right),$$

where here

$$\mathcal{L}^\dagger[u^0] = \left(c - 2u^0 - k^2 c u^0_{\theta\theta}\right) - 2k^2 c\partial_\theta \left(u^0_\theta \,\cdot\right) - k^2 c\partial_\theta^2 \left(u^0 \,\cdot\right)$$

denotes the adjoint operator of $\mathcal{L}[u^0]$ on $L^2_{\mathrm{per}}(0,1)$. In particular, noting the identity[6]

$$\mathcal{L}^\dagger[f]\left(f^{-3}g\right) = f^{-3}\mathcal{L}[f]g, \tag{2.3.5}$$

a straightforward calculation[7] shows that

$$\ker_{L^2_{\mathrm{per}}(0,1)} \left(\mathcal{L}^\dagger[u^0]\partial_\theta\right) = \operatorname{span}\left\{1, (u^0)^{-2}\right\}.$$

Thus, our two solvability conditions become

$$\left\langle 1, G[u^0]u_S^0 + F(u^0)\right\rangle_{L^2_{\mathrm{per}}(0,1;d\theta)} = 0 \quad \text{and} \quad \left\langle (u^0)^{-2}, G[u^0]u_S^0 + F(u^0)\right\rangle_{L^2_{\mathrm{per}}(0,1;d\theta)} = 0.$$

To put the above solvability conditions in a more useful form, we note that since

$$G^\dagger[u^0] = \left(1 + k^2 u^0_{\theta\theta}\right) - k^2\partial_\theta^2\left(u^0\cdot\right),$$

integration by parts (in $\theta$) and the identity $G^\dagger[u_0]1 = 1$ imply that the first equation above can be

[6]Although you can verify the identity using the forms listed above, this identity immediately follows from the alternative form for $\mathcal{L}[f]$ given in (2.4.3) below.

[7]Here, we are using that differentiating (2.3.2) with respect to $\theta$ gives $\mathcal{L}[u^0]u^0_\theta = 0$. Further, we note the third linearly independent solution of $\mathcal{L}^\dagger[u_0]v_\theta = 0$ is not periodic in $\theta$.

rewritten as

$$\partial_S M(u^0) = -\left\langle 1, ((u^0)^2)_X - 2(k^2)_X c(u^0_\theta)^2 - 2k^2 c_X (u^0_\theta)^2 - 2k^2 c((u^0_\theta)^2)_X \right\rangle$$
$$= \partial_X \left\langle 1, 2k^2 c(u^0_\theta)^2 - (u^0)^2 \right\rangle_{L^2_{\mathrm{per}}(0,1;d\theta)},$$

where here $M(u^0) = \int_0^1 u^0(\theta) d\theta$ is simply the conserved quantity $M$ in (2.2.2) evaluated at the 1-periodic traveling wave $u^0(\cdot)$. Similarly, using the identities

$$\left\langle (u^0)^{-2}, u^0 u^0_{\theta\theta} \right\rangle = \left\langle (u^0)^{-2}, (u^0_\theta)^2 \right\rangle \quad \left\langle (u^0)^{-2}, u^0 u^0_{X\theta\theta} \right\rangle = \left\langle (u^0)^{-2}, u^0_\theta u^0_{X\theta} \right\rangle,$$

which follow from integration by parts, the second solvability condition can be rewritten as

$$\left\langle G^\dagger[u^0](u^0)^{-2}, u^0_S \right\rangle = -\left\langle (u^0)^{-2}, ((u^0)^2)_X + \frac{3}{2}(k^2)_X c(u^0_\theta)^2 - \frac{1}{2}(k^2)_X c(u^0_\theta)^2 \right\rangle$$
$$- \left\langle (u^0)^{-2}, 2k^2 c_X (u^0_\theta)^2 + 2k^2 c u^0_\theta u^0_{X\theta} - k^2 c((u^0_\theta)^2)_X \right\rangle$$
$$= -\left\langle (u^0)^{-2}, ((u^0)^2)_X \right\rangle - 2k\,(kc)_X \left\langle (u^0)^{-2}, (u^0_\theta)^2 \right\rangle.$$

Using (2.3.3) and the fact that

$$\partial_S Q(u^0) = -\left\langle G^\dagger[u^0](u^0)^{-2}, u_S \right\rangle + 2k k_S \left\langle (u^0)^{-2}, (u^0_\theta)^2 \right\rangle,$$

where here $Q(u^0) = \int_0^1 \frac{u^0 + (u^0_\theta)^2}{(u^0)^2} d\theta$ is the conserved quantity $Q$ in (2.2.2) evaluated at the 1-periodic traveling wave $u^0(\cdot)$, the above can be rewritten as

$$\partial_S Q(u^0) = \left\langle (u^0)^{-2}, ((u^0)^2)_X \right\rangle = \partial_X \left\langle 1, 2\ln|u^0| \right\rangle.$$

Taken together, (2.3.3) and the above solvability conditions yield the first order, $3 \times 3$ system

$$\begin{cases} k_S + \partial_X(kc) = 0 \\ M_S = \partial_X \left\langle 1, 2k^2 c (u^0_\theta)^2 - (u^0)^2 \right\rangle_{L^2_{\text{per}}(0,1)} \\ Q_S = \partial_X \left\langle 1, 2 \ln |u^0| \right\rangle_{L^2_{\text{per}}(0,1)} \end{cases} \tag{2.3.6}$$

which, by the above formal arguments, is expected to govern (at least to leading order) the slow evolution of the wave number $k$ and the conserved quantities $M$ and $Q$ of a slow modulation of the periodic traveling wave $u^0$.

System (2.3.6) is referred to as the Whitham modulation system associated to the conduit equation (2.1.1). Heuristically, it is expected that the Whitham modulation equations (2.3.6) relate to the dynamical stability of periodic traveling wave solutions of (2.1.1) in the following way. Suppose $(a_0, E_0, c_0) \in \mathcal{B}$ so that $\phi(\cdot; a_0, E_0, c_0)$ is an even, $T = 1/k$-periodic solution of (2.2.4). From the above formal analysis, we see that (2.2.4) has a modulated periodic traveling wave of the form

$$u(x,t;\varepsilon) = \phi\left(\tfrac{1}{\varepsilon}\psi(\varepsilon x, \varepsilon t); a(\varepsilon x, \varepsilon t), E(\varepsilon x, \varepsilon t), c(\varepsilon x, \varepsilon t)\right) + o(\varepsilon),$$

where the parameters $(a(\varepsilon x, \varepsilon t), E(\varepsilon x, \varepsilon t), c(\varepsilon x, \varepsilon t))$ evolve near $(a_0, E_0, c_0)$ in $\mathcal{B}$ in such a way that $k$, $M$, and $Q$ satisfy the Whitham system (2.3.6). Note this requires that the nonlinear mapping

$$\mathbb{R}^3 \ni (a, E, c) \mapsto (k, M, Q) \in \mathbb{R}^3$$

be locally invertible near $(a_0, E_0, c_0)$. By the implicit function theorem, it is sufficient to assume the Jacobian of this mapping at $(a_0, E_0, c_0)$ is non-zero[8]. In particular, any 1-periodic solution $\phi_0$ of (2.2.4), being independent of the slow variables $X$ and $S$, is necessarily a constant solution of (2.3.6). The stability of $\phi_0$ to slow modulations is thus expected to be governed (to leading order, at least) by the linearization of (2.3.6) about $\phi_0$. Specifically, using the chain rule to rewrite (2.3.6)

[8]This condition will appear later in our rigorous theory as well: see the discussion following the proof of Theorem 2.4.6 below. It also appears in the formal work of Maiden & Hoefer [47].

in the quasilinear form

$$\begin{pmatrix} k \\ M \\ Q \end{pmatrix}_S = \mathbf{D}(u^0) \begin{pmatrix} k \\ M \\ Q \end{pmatrix}_X, \tag{2.3.7}$$

where here

$$\mathbf{D}(u) = \begin{pmatrix} -(c + kc_k) & -kc_M & -kc_Q \\ \left\langle 1, 2k^2 c u_\theta^2 - u^2 \right\rangle_k & \left\langle 1, 2k^2 c u_\theta^2 - u^2 \right\rangle_M & \left\langle 1, 2k^2 c u_\theta^2 - u^2 \right\rangle_Q \\ \langle 1, 2\ln|u| \rangle_k & \langle 1, 2\ln|u| \rangle_M & \langle 1, 2\ln|u| \rangle_Q \end{pmatrix},$$

it is natural to expect the stability of $\phi_0$ to slow modulations to be governed by the eigenvalues of the $3 \times 3$ matrix $\mathbf{D}(\phi_0)$. Indeed, linearizing (2.3.7) about the constant solution $\phi_0$, we see that the eigenvalues of the linearization are of the form

$$\widetilde{\lambda}_j(\xi) = i\xi\alpha_j,$$

where $\{\alpha_j\}_{j=1}^3$ are the eigenvalues of $\mathbf{D}(\phi_0)$ and $\xi \in \mathbb{R}$. Consequently, if the Whitham system is weakly hyperbolic[9] at $\phi_0$, so that the eigenvalues of $\mathbf{D}(\phi_0)$ are all real, then the eigenvalues of the linearization of (2.3.6) are purely imaginary, indicating a marginal (spectral) stability. Conversely, if $\mathbf{D}(\phi_0)$ has an eigenvalue with non-zero imaginary part, in which case (2.3.7) is elliptic at $\phi_0$, then the linearization of (2.3.6) has eigenvalues with positive real part, indicating (spectral) instability of $\phi_0$.

The goal of this chapter is to rigorously validate the above predictions of Whitham modulation theory as they pertain to the stability of periodic traveling wave solutions of (2.1.1). Following the works in [38, 5, 3], this will be accomplished by using rigorous spectral perturbation theory to analyze the spectrum of the linearization of (2.2.3) about such a solution and, in particular,

[9]Note full hyperbolicity of the system additionally requires the eigenvalues $\alpha_j$ are semi-simple, i.e. that their algebraic and geometric multiplicities agree.

relating the spectrum of the linearization in a neighborhood of the origin in the spectral plane to the eigenvalues of the matrix $\mathbf{D}(\phi_0)$ defined above. Our main result is the following, which establishes that weak-hyperbolicity of the Whitham system (2.3.6) is indeed a necessary condition for the spectral stability of the underlying wave $\phi_0$.

**Theorem 2.3.1.** *Suppose $\phi_0$ is an even, $T_0 = 1/k_0$-periodic, strictly positive traveling wave solution of* (2.1.1) *with wave speed $c_0 > 0$, and that the set of nearby periodic traveling wave profiles $\phi$ with speed close to $c_0$ is a* 3*-dimensional manifold parameterized by $(k, M(\phi), Q(\phi))$, where $1/k$ denotes the fundamental period of $\phi$. Then a necessary condition for $\phi_0$ to be spectrally stable is that the Whitham modulation system* (2.3.6) *be weakly hyperbolic at $(k_0, M(\phi_0), Q(\phi_0))$, in the sense that all their characteristic speeds must be real.*

To prove Theorem 2.3.1 we will show that, under appropriate non-degeneracy assumptions, the spectrum of the linearization of (2.2.3) about a periodic traveling wave $\phi_0$ consists, in a sufficiently small neighborhood of the origin, of precisely three $C^1$ curves which expand as

$$\lambda_j(\xi) = \widetilde{\lambda}_j(\xi) + O(\xi) = i\alpha_j\xi + O(\xi), \quad |\xi| \ll 1,$$

where here the $\alpha_j$ are precisely the eigenvalues of the matrix $\mathbf{D}(\phi_0)$. Interestingly, this shows that spectral stability in a neighborhood of the origin of $\phi_0$, otherwise known as "modulational stability" (see Definition 1.3.2), cannot be concluded from the weak, or even strong, hyperbolicity of the Whitham modulation system (2.3.6). While we do not pursue it here, such information may be able to be deduced from determining the second-order corrector in $\xi$ to the spectral curves $\lambda_j(\xi)$ deduced above.

## 2.4 Rigorous Modulational Stability Theory

We now begin our rigorous mathematical study of the dynamical stability of periodic traveling wave solutions of (2.1.1) when subject to localized, i.e., integrable, perturbations on the line.

Following [38, 5, 3], we conduct a detailed analysis of the spectral problem associated with the linearization of (2.1.1) about a periodic traveling wave solution. The first step in this analysis is to understand the structure of the generalized kernel of the associated linearized operators when subject to perturbations that are co-periodic with the underlying wave. With this information in hand, we then use Floquet-Bloch theory and rigorous spectral perturbation theory to obtain an asymptotic description of the spectrum of the linearization considered as an operator on $L^2(\mathbb{R})$ in a sufficiently small neighborhood of the origin.

### 2.4.1 Linearization & Set Up

To begin, let $(a, E, c) \in \mathcal{B}$ and denote by $\phi = \phi(\cdot; a, E, c)$ the corresponding even, $T = T(a, E, c)$-periodic equilibrium solution of (2.2.3). We are now interested in rigorously describing the local dynamics of (2.2.3) near $\phi$. Specifically, we are interested in understanding if $\phi$ is stable to small *localized*, i.e., integrable on $\mathbb{R}$, perturbations. To this end, we note that the linearization of (2.2.3) about $\phi$ is given by

$$G[\phi]v_t = \partial_z \mathcal{L}[\phi]v, \tag{2.4.1}$$

where here $G[\phi]$ and $\mathcal{L}[\phi]$ are linear operators on $L^2(\mathbb{R})$ defined by

$$\begin{cases} G[\phi]f := f - (\phi^2(\phi^{-1}f)')' = f + \phi'' f - \phi f'', \\ \mathcal{L}[\phi]f := cf - 2\phi f - c\phi'' f + 2c\phi' f' - c\phi f''. \end{cases} \tag{2.4.2}$$

Note that $\mathcal{L}[\phi]$ can also be written in the form

$$\mathcal{L}[\phi]f = cf - 2\phi f - 2c\phi(\phi^{-1}\phi')' f - c\phi^2(\phi^{-1}f)''. \tag{2.4.3}$$

Observe these are both closed linear operators on $L^2(\mathbb{R})$ with densely defined domains $H^2(\mathbb{R})$. As the linear evolution equation (2.4.1) is autonomous in time, its dynamics can be (at least partly) understood by studying the associated generalized spectral problem

$$\lambda G[\phi]v = \partial_z \mathcal{L}[\phi]v, \tag{2.4.4}$$

posed on $L^2(\mathbb{R})$, where here $\lambda \in \mathbb{C}$ is a spectral parameter corresponding to the temporal frequency of the perturbation. To put (2.4.4) in a more standard form, we note the following lemma.

**Lemma 2.4.1.** *The operator $G[\phi] : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ defined in* (2.4.2)*(i) is a (weakly) invertible operator. That is, for every $g \in L^2(\mathbb{R})$ the equation*

$$G[\phi]v = g$$

*has a unique weak solution in $H^1(\mathbb{R})$.*

*Proof.* Observe that by defining

$$H[\phi]f := G[\phi](\phi f) = \phi f - (\phi^2 f')'$$

we have $H[\phi](\phi^{-1}f) = G[\phi]f$ so that, in particular, $G[\phi]$ is (weakly) invertible if and only if $H[\phi]$ is (weakly) invertible. Since $\phi > 0$ uniformly, it follows that $H[\phi]$ is a symmetric, uniformly elliptic differential operator. Consequently, a standard argument using the Riesz representation theorem implies that for every $g \in L^2(\mathbb{R})$ the elliptic equation

$$H[\phi]f = g$$

has a unique weak solution in $H^1(\mathbb{R})$, i.e. that $H[\phi] : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is (weakly) invertible. The result now follows. □

**Remark 2.4.2.** *The (weak) invertibility of $G[\phi]$ implies that the bilinear form generated by*[10]

[10]Throughout, we denote, with a slight abuse of notation, the operator $(G[\phi])^{-1}$ by simply $G^{-1}[\phi]$. The same abuse of notation will be used when referring to adjoints of operators depending on $\phi$.

*$G^{-1}[\phi]$ is well-defined on $L^2(\mathbb{R})$. As we will see, this will be sufficient in order to verify Theorem 2.3.1.*

By Lemma 2.4.1, the generalized spectral problem (2.4.4) is equivalent to the spectral problem for the linear operator

$$A[\phi] := G^{-1}[\phi]\partial_z \mathcal{L}[\phi] \tag{2.4.5}$$

considered as a closed, densely defined linear operator on $L^2(\mathbb{R})$. Motivated by the above considerations, we recall Definition 1.2.6 and say that a periodic traveling wave $\phi$ of (2.1.1) is said to be *spectrally unstable* if the $L^2(\mathbb{R})$-spectrum of $A[\phi]$ intersects the open right half plane, i.e. if

$$\sigma_{L^2(\mathbb{R})}(A[\phi]) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} \neq \emptyset,$$

while it is *spectrally stable* otherwise. This motivates a detailed study of the spectrum of the linear operator $A[\phi]$.

**Remark 2.4.3.** *Observe that while* (2.1.1) *is a nonlinear dispersive PDE, it does not possess a Hamiltonian structure. Consequently, while the spectrum of $A[\phi]$ is symmetric about the real axis, owing to the fact that $\phi$ is real-valued, it is not necessarily symmetric about the imaginary axis.*

**Remark 2.4.4.** *Recall that in [65] the authors considered the stability of solitary traveling wave solutions of the magma equations* (2.1.2)*, which corresponds to the conduit equation* (2.1.1) *when* $(n, m) = (2, 1)$*. In that case, the linearization (in [65]) of* (2.2.3) *about a solitary wave $\phi$ is given, after some manipulation, by*

$$G[\phi]v_t = v_t - (\phi^2(\phi^{-1}v_t)_z)_z = \partial_z\left[-cv + 2(c-1)\phi^{-1}v - c\phi^2\partial_z^2(\phi^{-1}v)\right]$$

*which, using* (2.4.3)*, is equivalent to our representation* (2.4.4) *provided that $\phi$ satisfies*

$$\phi + c\phi(\phi^{-1}\phi')' = c + (1-c)\phi^{-1}$$

*Recalling Remark 2.2.2, this latter condition follows directly from* (2.2.4) *with the choice* $2cE = c-1$ *associated to the solitary waves considered in [65].*

Recall that we have already characterized the $L^2(\mathbb{R})$-spectrum of $T$-periodic coefficient linear differential operators in Section 1.3.1. In particular, we demonstrated that the $L^2(\mathbb{R})$-spectrum satisfies the spectral decomposition (1.3.6). While $A[\phi]$ is not in the form that we studied in Section 1.3.1, it can be shown that the $L^2(\mathbb{R})$-spectrum of $A[\phi]$ still satisfies the spectral decomposition

$$\sigma_{L^2(\mathbb{R})}\left(A[\phi]\right) = \bigcup_{\xi\in[-\pi/T,\pi/T)} \sigma_{L^2_{\text{per}}(0,T)}\left(A_\xi[\phi]\right),$$

thereby continuously parameterizing the essential $L^2(\mathbb{R})$-spectrum of $A[\phi]$ by a one-parameter family of $T$-periodic eigenvalues of the associated Bloch operators $A_\xi[\phi] := e^{-i\xi x}A[\phi]e^{i\xi x}$. For more details on this extension beyond differential operators, see [20, 28].

To determine the spectral stability of a periodic traveling wave $\phi$, one must therefore determine all of the $T$-periodic eigenvalues for each Bloch operator for $\xi \in [-\pi/T, \pi/T)$. Outside of some very special cases, one does not expect to be able to do this complete spectral analysis explicitly. However, as we saw in the discussion following Definition 1.3.2, for the purposes of modulational stability analysis, we need only consider the spectrum of the operators $A_\xi[\phi]$ in a neighborhood of the origin in the spectral plane and only for $|\xi| \ll 1$. Recalling this discussion, observe from (1.3.4) – which, again, still holds for our present case – that the spectrum of $A_0[\phi]$ corresponds to the spectral stability of $\phi$ to $T$-periodic perturbations, i.e. to perturbations with the same period as the carrier wave. Similarly, $|\xi| \ll 1$ corresponds to long wavelength perturbations of the carrier wave. Furthermore, slow modulations of $\phi$ form a special class of long wavelength perturbations in which the effect of the perturbation is to slowly vary, namely modulate, the wave characteristics – the parameters $a$, $E$ and $c$ in the present setting – and the translational mode. As we will see, variations in these parameters naturally provide spectral information about the co-periodic Bloch operator $A_0[\phi]$ at the origin in the spectral plane. From the above considerations, it is natural to expect that the spectral stability of the underlying wave $\phi$ to slow modulations corresponds to the case when the spectrum of the Bloch operators $A_\xi[\phi]$ near $(\lambda, \xi) = (0,0)$ lie in the closed left half-plane. For more discussion regarding this motivation, see [5].

In order to prove Theorem 2.3.1, our program will roughly break down into three steps. First, we will analyze the structure of the generalized kernel of the unmodulated Bloch operator $A_0[\phi]$, showing that, under certain geometric conditions, this operator has $\lambda = 0$ as an eigenvalue with algebraic multiplicity three and geometric multiplicity two. Secondly, we will use rigorous spectral perturbation theory to examine how the spectrum near the origin of the modulated operators $A_\xi[\phi]$ bifurcates from $\lambda = 0$ for $0 < |\xi| \ll 1$. Through this, we will derive a $3 \times 3$ linear system that encodes the leading order asymptotics of the spectral curves near $\lambda = 0$ for $0 < |\xi| \ll 1$. Finally, we will see by a direct term by term comparison that this linear system, derived through rigorous spectral perturbation theory, agrees *exactly* (up to a harmless shift by the identity) with the linearized Whitham modulation system (2.3.6).

### 2.4.2 Analysis of the Unmodulated Operators

As described above, the first main step in our analysis is to understand the $T$-periodic generalized kernel of the unmodulated operator $A_0[\phi]$ defined in (2.4.5), as well as its adjoint operator[11] $A_0^\dagger[\phi]$. We begin by characterizing the $T$-periodic kernel of $\mathcal{L}[\phi]$ and its adjoint. To this end, note that differentiating the profile equation (2.2.4) with respect to $z$, as well as with respect to the parameters $a$, $E$, and $c$, yields the identities

$$\mathcal{L}[\phi]\phi' = 0 = \mathcal{L}[\phi]\phi_a, \quad \mathcal{L}[\phi]\phi_E = 2c, \quad \mathcal{L}[\phi]\phi_c = 2E - \left(\phi + (\phi')^2 - \phi\phi''\right). \tag{2.4.6}$$

Recalling that $\mathcal{L}[\phi]$ and $\mathcal{L}^\dagger[\phi]$ are related via (2.3.5), we have the following result.

**Lemma 2.4.5.** *Let $\phi$ be a non-trivial $T$-periodic solution of the profile equation* (2.2.4). *So long*

[11] Information about the adjoint is necessary to construct the spectral projections for $A_0[\phi]$ at $\lambda = 0$.

*as $T_a \neq 0$, we have*

$$\ker_{L^2_{\text{per}}(0,T)}(\mathcal{L}[\phi]) = \text{span}\{\phi'\} \quad \text{and} \quad \ker_{L^2_{\text{per}}(0,T)}\left(\mathcal{L}^\dagger[\phi]\right) = \text{span}\left\{\phi^{-3}\phi'\right\}.$$

*Under the same assumption, we also have*

$$\ker_{L^2_{\text{per}}(0,T)}\left(A^\dagger[\phi]\right) = \text{span}\left\{1, G^\dagger[\phi]\phi^{-2}\right\}.$$

*Proof.* Recalling $\phi$ may be chosen to be even, we have that $\phi'$ and $\phi_a$ are odd and even functions of $z$, respectively, so it follows from (2.4.6) that $\phi'$ and $\phi_a$ provide two linearly independent solutions of the second order differential equation $\mathcal{L}[\phi]f = 0$. To identify the kernel of $\mathcal{L}[\phi]$, we must impose $T$-periodic boundary conditions. Since $\phi'$ is clearly $T$-periodic, it follows that the $T$-periodic kernel of $\mathcal{L}[\phi]$ has dimension at least one. However, the fact that $T$ depends on the parameter $a$ implies that the function $u_a$ is generally not periodic. Indeed, differentiating the obvious relation

$$\begin{pmatrix} \phi(0; a, E, c) \\ \phi'(0; a, E, c) \end{pmatrix} = \begin{pmatrix} \phi(T; a, E, c) \\ \phi'(T; a, E, c) \end{pmatrix}$$

with respect to $a$ gives

$$\begin{pmatrix} \phi_a(0) \\ \phi_{ax}(0) \end{pmatrix} - \begin{pmatrix} \phi_a(T) \\ \phi_{ax}(T) \end{pmatrix} = T_a \begin{pmatrix} \phi'(T) \\ \phi''(T) \end{pmatrix}$$

which, using that $\mathcal{L}[\phi]$ is a second order differential equation and that $\phi$ is non-trivial[12], implies that $\phi_a$ is $T$-periodic if and only if $T_a$ is zero. This yields the characterization of the $T$-periodic kernel of $\mathcal{L}[\phi]$, and the kernel of $\mathcal{L}^\dagger[\phi]$ follows immediately from (2.3.5).

Finally, to characterize the $T$-periodic kernel of $A^\dagger[\phi]$, observe that

$$A^\dagger[\phi] = -\mathcal{L}^\dagger[\phi]\partial_x(G^{-1})^\dagger[\phi].$$

[12]Note since $\phi$ satisfies the second order ODE (2.2.4), vanishing of the vector $(\phi'(T), \phi''(T))^T$ would imply $\phi$ is the the trivial solution by uniqueness. Alternatively, note that while $\phi'(T) = 0$ by normalization, a direct calculation from (2.2.5) shows that $\phi''(T) = -V_\phi(\phi_{\min}; a, c)$, which again is non-zero since $\phi$ is not an equilibrium solution of (2.2.4).

Since the adjoint of $G[\phi]$ is invertible on the space of $T$-periodic functions, and recalling $G^\dagger[\phi]1 = 1$, the claim now follows from the characterization of the kernel of $\mathcal{L}^\dagger[\phi]$. $\square$

Next, we use Lemma 2.4.5 along with the Fredholm alternative to identify, under appropriate genericity conditions, the $T$-periodic generalized kernels of $A[\phi]$ and $A^\dagger[\phi]$. To this end, observe that (2.4.6) implies that

$$A[\phi]\{\phi', \phi_a, \phi_E\} = 0 \text{ and } A[\phi]\phi_c = -\phi',$$

which, among other things, yields three linearly independent functions satisfying the third order ODE $\partial_x \mathcal{L}[\phi]f = 0$. In Lemma 2.4.5 above, we showed that $\phi_a$ is not $T$-periodic provided that $T_a$ is non-zero. Using similar arguments, it is readily seen that the functions $\phi_E$ and $\phi_c$ are not $T$-periodic provided that $T_E$ and $T_c$ are non-zero, respectively. Indeed, we find that

$$\begin{pmatrix} \phi_E(0) \\ \phi_{Ex}(0) \\ \phi_{Exx}(0) \end{pmatrix} - \begin{pmatrix} \phi_E(T) \\ \phi_{Ex}(T) \\ \phi_{Exx}(T) \end{pmatrix} = T_E \begin{pmatrix} \phi'(T) \\ \phi''(T) \\ \phi'''(T) \end{pmatrix},$$

with an analogous equation holding for $\phi_c$. Recalling by the above discussion that $\phi''(T)$ is non-zero, the desired result follows. For notational simplicity, we introduce the following Poisson bracket style notation for two-by-two Jacobian determinants

$$\{F, G\}_{x,y} := \det \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

and an analogous notation for three-by three Jacobian determinants:

$$\{F, G, H\}_{x,y,z} := \det \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix}.$$

Using the above identities, it follows that, while $\phi_a$ and $\phi_E$ are not individually $T$-periodic, the linear combination

$$T_a \phi_E - T_E \phi_a = \{T, \phi\}_{a,E}$$

lies in the $T$-periodic kernel of $A[\phi]$. Similarly, we see that the functions $\{T, \phi\}_{a,c}$ and $\{T, \phi\}_{E,c}$ are both $T$-periodic and satisfy

$$A[\phi]\{T, \phi\}_{a,c} = -T_a \phi' \quad \text{and} \quad A[\phi]\{T, \phi\}_{E,c} = -T_E \phi'. \tag{2.4.7}$$

We now state the main result for this section.

**Theorem 2.4.6.** *Let $\phi = \phi(\cdot; a, E, c)$ be a $T$-periodic solution of the profile equation* (2.2.4)*, and assume the Jacobians $T_a$, $\{T, M\}_{a,E}$ and $\{T, M, Q\}_{a,E,c}$ are non-zero. Then $\lambda = 0$ is an eigenvalue of the Bloch operator $A_0[\phi]$ with algebraic multiplicity three and geometric multiplicity two. In particular, defining*

$$\begin{aligned} &\Phi_1 := \{T, M\}_{a,E}\phi' && \Phi_2 := \{T, \phi\}_{a,E} && \Phi_3 := \{T, M, \phi\}_{a,E,c} \\ &\Psi_1 := \eta && \Psi_2 := 1 && \Psi_3 := -\{T, M\}_{a,E} G^\dagger[\phi]\phi^{-2} - \{T, Q\}_{a,E}, \end{aligned}$$

*where $\eta \in L^2_{\text{per}}(0, T)$ is the unique odd function satisfying $A_0^\dagger[\phi]\eta = -\Psi_3$, we have that $\{\Phi_\ell\}_{\ell=1}^3$ and $\{\Psi_j\}_{j=1}^3$ provide a biorthogonal bases for the generalized kernels of of $A_0[\phi]$ and $A_0^\dagger[\phi]$, respectively. In particular, we have $\langle \Psi_j, \Phi_\ell \rangle = 0$ if and only if $j \neq \ell$. Furthermore, the functions $\Phi_\ell$ and $\Psi_j$ satisfy the equations*

$$A_0[\phi]\Phi_1 = 0 = A_0[\phi]\Phi_2, \;\; A_0[\phi]\Phi_3 = -\Phi_1,$$

*and*

$$A_0^\dagger[\phi]\Psi_2 = 0 = A_0^\dagger[\phi]\Psi_3, \;\; A_0^\dagger[\phi]\Psi_1 = -\Psi_3.$$

*Proof.* Since $T_a \neq 0$ by assumption, the characterization of the kernel of $A_0^\dagger[\phi]$ follows from

Lemma 2.4.5. Further, note a function $f$ belongs to the $T$-periodic kernel of $A_0[\phi]$ if and only if $f$ is $T$-periodic and either $\mathcal{L}[\phi]f = 0$ or $\mathcal{L}[\phi]f$ is a non-zero constant. From (2.4.6), it follows immediately that

$$\ker_{L^2_{\text{per}}(0,T)}\left(A_0[\phi]\right) = \text{span}\left\{\Phi_1, \Phi_2\right\}.$$

Furthermore, $\phi'$ is in the range of $A_0[\phi]$ by (2.4.7). Hence, by the Fredholm alternative (or by parity), we have that $\langle\Psi_2, \Phi_1\rangle = 0 = \langle\Psi_3, \Phi_1\rangle$. For the periodic element that lies in the Jordan chain above $\phi'$, we take a specific linear combination of $\{T, \phi\}_{a,c}$ and $\{T, \phi\}_{E,c}$, namely $\Phi_3$. Furthermore, by the Fredholm alternative the Jordan chain above $\phi'$ terminates at height one if and only if

$$\Phi_3 \notin \ker\left(A_0^\dagger[\phi]\right)^\perp.$$

Since $\Phi_3$ has zero mean by construction, we clearly have $\langle\Psi_2, \Phi_3\rangle = 0$ and hence the above condition is equivalent to showing

$$\langle\Psi_3, \Phi_3\rangle = -\{T, M\}_{a,E}\left\langle G^\dagger[\phi]\phi^{-2}, \{T, M, \phi\}_{a,E,c}\right\rangle$$

is non-zero. To write the above in a more usable form, observe from the definition of $Q$ in (2.2.6) that

$$Q_a = \int_0^T \left(\frac{2\phi\phi'\phi'_a - \phi\phi_a - 2(\phi')^2\phi_a}{\phi^3}\right) dx + \frac{[\phi(T) + (\phi'(T))^2]T_a}{(\phi(T))^2}.$$

Note by integration by parts that

$$\int_0^T \left(\frac{\phi'}{\phi^2}\right)\phi'_a dx = -\int_0^T \left(\frac{\phi\phi'' - 2(\phi')^2}{\phi^3}\right)\phi_a\, dx$$

and hence

$$\begin{aligned} Q_a &= -\int_0^T \left(\frac{\phi + 2\phi\phi'' - 2(\phi')^2}{\phi^3}\right)\phi_a\, dx + \frac{[\phi(T) + (\phi'(T))^2]T_a}{(\phi(T))^2} \\ &= -\left\langle G^\dagger[\phi]\phi^{-2}, \phi_a\right\rangle + \frac{[\phi(T) + (\phi'(T))^2]T_a}{(\phi(T))^2}. \end{aligned} \tag{2.4.8}$$

Similar expressions hold for $Q_E$ and $Q_c$ and hence we find

$$\langle \Psi_3, \Phi_3 \rangle = \{T, M\}_{a,E}\{T, M, Q\}_{a,E,c},$$

which is non-zero by hypotheses. The proves our characterization of the generalized $T$-periodic kernel of the operator $A_0[\phi]$.

Finally, we consider the generalized kernel of the adjoint operator $A_0^\dagger[\phi]$. Following the method for calculating $\langle \Psi_3, \Phi_3 \rangle$ above, we immediately find that $\langle \Psi_2, \Phi_2 \rangle = \{T, M\}_{a,E}$, which is assumed to be non-zero. Hence, by the Fredholm alternative, $\Psi_2$ is not in the range of $A_0^\dagger[\phi]$. By similar arguments, we find

$$\langle \Psi_3, \Phi_2 \rangle = -\{T, M\}_{a,E} \left( \left\langle G^\dagger[\phi]\phi^{-2}, \{T, \phi\}_{a,E} \right\rangle + \{T, Q\}_{a,E} \right) = 0,$$

so that, again by the Fredholm alternative, $\Psi_3$ belongs in the range of $A_0^\dagger[\phi]$. Since $\Phi_3$ is even and $A_0[\phi]$ switches parity, the fact that the kernel of $A_0[\phi]$ consists entirely of even functions implies there exists a unique $T$-periodic odd function $\eta$ that satisfies

$$A_0^\dagger[\phi]\eta = -\Psi_3.$$

Furthermore, we note that $\Psi_1 = \eta$ is not in the range of $A_0^\dagger[\phi]$ since

$$\langle \Psi_1, \Phi_1 \rangle = -\langle \Psi_1, A_0[\phi]\Phi_3 \rangle = \langle \Psi_3, \Phi_3 \rangle = \{T, M\}_{a,E}\{T, M, Q\}_{a,E,c},$$

which is non-zero by assumption. This completes the characterization of the generalized kernel of $A_0^\dagger[\phi]$. To finish the proof, we note that $\langle \Psi_1, \Phi_2 \rangle = 0 = \langle \Psi_1, \Phi_3 \rangle$ by parity. □

We now make some important comments regarding the assumptions in Theorem 2.4.6. The above result was obtained through the observation that infinitesimal variations along the manifold of periodic traveling wave solutions yield tangent vectors that lie in the generalized kernels. Through our existence theory in Section 2.2.2, this manifold was parameterized by the wave speed $c$ and the integration constants $a$ and $E$. While this parameterization is natural from the mathematical perspective, following directly from the Hamiltonian formulation (2.2.5) of the profile equation (2.2.4), it is different than the parameteriztaion that naturally arises in Whitham theory. Indeed, recall from Section 2.3 that the Whitham modulation system (2.3.6) describes the slow evolution of the wave number $k$ and conserved quantities $M$ and $Q$, thus yielding a parameterization of the manifold of periodic traveling wave solutions of the conduit equation (2.1.1) in terms of these physical quantities. Consequently, a-priori these two approaches work with different parameterizations of the same manifold.

In order to make comparisons between these two approaches, it is natural to assume that we can smoothly change between these parameterizations. Specifically, we require that the manifold of periodic traveling wave solutions of (2.1.1) constructed in Section 2.2.2 can be locally reparameterized in a $C^1$ manner by the wave number $k$ and the conserved quantities $M$ and $Q$, i.e. that the mapping

$$\mathcal{B} \ni (a, E, c) \mapsto (k(a, E, c), M(a, E, c), Q(a, E, c)) \in \mathbb{R}^3$$

is locally $C^1$-invertible at each point. By the Implicit Function Theorem, this is guaranteed by requiring that the Jacobian determinant

$$\frac{\partial(k, M, Q)}{\partial(a, E, c)} = \{k, M, Q\}_{a,E,c}$$

of the above map is non-singular at each point in $\mathcal{B}$. Recalling that $k(a, E, c) = \frac{1}{T(a,E,c)}$ we see that

$$\{k, M, Q\}_{a,E,c} = -\frac{1}{T^2} \{T, M, Q\}_{a,E,c},$$

it follows that such a $C^1$-reparameterization is possible provided $\{T, M, Q\}_{a,E,c} \neq 0$, which is one of the primary assumptions in Theorem 2.4.6. Likewise, the requirement that $\{T, M\}_{a,E} \neq 0$ is equivalent to saying that waves with fixed wave speed can be locally reparameterized in a $C^1$

manner by the wave number $k$ and mass $M$.

With the above observations in mind, we now seek a restatement of Theorem 2.4.6 that is generated with respect to infinitesimal variations along the $k$, $M$, and $Q$ coordinates. To see the dependence on the wave number $k$ explicitly, we begin by rescaling the spatial variable as $y = kx$ and note that $T$-periodic traveling wave solutions of (2.1.1) correspond to 1-periodic traveling wave solutions (with the same period) of the rescaled evolution equation

$$u_t + k(u^2)_y - k^2(u^2(u^{-1}u_t)_y)_y = 0 \tag{2.4.9}$$

where $k = 1/T$. After rescaling the traveling coordinate $\theta = kz = k(x - ct) = y - \omega t$, where $\omega = kc$ is temporal frequency, it is readily seen that traveling wave solutions of (2.4.9) correspond to solutions of the form $u(y,t) = \phi(y - \omega t)$. Hence, $\phi(\theta)$ is a stationary 1-periodic solution of the evolutionary equation

$$u_t - kcu_\theta + k(u^2)_\theta - k^2(u^2(u^{-1}(u_t - kcu_\theta)_\theta)_\theta = 0 \tag{2.4.10}$$

The rescaled profile equation now reads

$$2cE = c\phi - \phi^2 + k^2c(\phi')^2 - k^2c\phi\phi'', \tag{2.4.11}$$

where now $'$ denotes differentiation with respect to $\theta$. Through this rescaling, the $T$-periodic solutions $\phi$ of (2.2.4) now correspond to 1-periodic solutions of (2.4.11) for some $k$. Similarly, the conserved quantities evaluated on the manifold of 1-periodic solutions of (2.4.11) take the form

$$\begin{cases} M := \displaystyle\int_0^1 \phi(\theta; k, M, Q)\, d\theta \\ Q := \displaystyle\int_0^1 \frac{\phi(\theta; k, M, Q) + k^2(\phi'(\theta; k, M, Q))^2}{(\phi(\theta; k, M, Q))^2}\, d\theta \end{cases} \tag{2.4.12}$$

while the linearization of (2.4.10) now reads

$$G[\phi]v_t = k\partial_\theta \mathcal{L}[\phi]v$$

where, with a slight abuse of notation, $G[\phi]$ and $\mathcal{L}[\phi]$ are defined as in (2.4.2), albeit with $k\partial_\theta$ replacing $\partial_z$, i.e.

$$\begin{cases} G[\phi]f = f - k^2(\phi^2(\phi^{-1}f)')' = f + k^2\phi'' f - k^2\phi f'' \\ \mathcal{L}[\phi]f = cf - 2\phi f - k^2 c\phi'' f + 2k^2 c\phi' f' - k^2 c\phi f'' \end{cases} \tag{2.4.13}$$

Clearly, the rescaled operator $G[\phi]$ is invertible as before, leading us to the consideration of the spectral problem

$$A[\phi]v = \lambda v \tag{2.4.14}$$

posed on $L^2(\mathbb{R})$, where now, again with a slight abuse of notation, $A[\phi] = G^{-1}[\phi]k\partial_\theta\mathcal{L}[\phi]$ is a linear operator with 1-periodic coefficients and $G[\phi]$ and $\mathcal{L}[\phi]$ are differential operators as in (2.4.13).

By Theorem 2.4.6, we know that $\lambda = 0$ is a 1-periodic eigenvalue of the rescaled operator $A[\phi]$ in (2.4.14) with algebraic multiplicity three and geometric multiplicity two. In fact, differentiating (2.4.11) with respect to $x$, $M$, and $Q$ yields

$$A[\phi]\phi' = 0, \quad A[\phi]\phi_M = -kc_M\phi', \quad A[\phi]\phi_Q = -kc_Q\phi'$$

Observe that since the profiles $\phi(\cdot; k, M, Q)$ are always 1-periodic by construction, it follows the functions $\phi_M$ and $\phi_Q$ are also 1-periodic. In particular, the functions $\phi'$ and $c_M\phi_Q - c_Q\phi_M$ are linearly independent and span the 1-periodic kernel of $A[\phi]$. We also mention that, as in Lemma 2.4.5, the functions 1 and $G^\dagger[\phi]\phi^{-2}$ span the 1-periodic kernel of the rescaled operator $A^\dagger[\phi]$. Furthermore, using similar arguments as in (2.4.8) we see that

$$\langle 1, \phi_M\rangle = 1 = -\left\langle G^\dagger[\phi]\phi^{-2}, \phi_Q\right\rangle$$

and

$$\left\langle 1, \phi_Q \right\rangle = 0 = \left\langle G^\dagger[\phi]\phi^{-2}, \phi_M \right\rangle.$$

We also have

$$\langle 1, \phi' \rangle = 0 = \left\langle G^\dagger[\phi]\phi^{-2}, \phi' \right\rangle$$

by parity. Consequently, there is a linear combination of the functions 1 and $G^\dagger[\phi]\phi^{-2}$ that is orthogonal to the 1-periodic kernel of $A[\phi]$. It follows there is a 1-periodic function in the generalized kernel of $A^\dagger[\phi]$. Letting $A_\xi[\phi]$ denote the Bloch operators associated with $A[\phi]$, defined now for $\xi \in [-\pi, \pi)$, we have the following result.

**Corollary 2.4.7.** *Let $\phi = \phi(a, E, c)$ be a $T$-periodic solution of the profile equation* (2.2.4)*, and assume that the Jacobian determinants $T_a$, $\{T, M\}_{a,E}$ and $\{T, M, Q\}_{a,E,c}$ are all non-zero. Then $\lambda = 0$ is an eigenvalue of $A_0[\phi]$ with algebraic multiplicity three and geometric multiplicity two. In particular, defining*

$$\begin{aligned} &\Phi_1^0 := \phi' \qquad \Phi_2^0 := \phi_M \qquad \Phi_3^0 := \phi_Q \\ &\Psi_1^0 := \beta \qquad \Psi_2^0 := 1 \qquad \Psi_3^0 := -G^\dagger[\phi]\phi^{-2}, \end{aligned}$$

*where $\beta \in L^2_{\mathrm{per}}(0,1)$ is the unique odd function satisfying $A_0^\dagger[\phi]\beta \in \ker\left(A_0^\dagger[\phi]\right)$ and $\left\langle \beta, \Phi_1^0 \right\rangle = 1$, we have that $\{\Phi_\ell^0\}_{\ell=1}^3$ and $\{\Psi_j^0\}_{j=1}^3$ provide a basis of solutions for the generalized kernels of $A_0[\phi]$ and $A_0^\dagger[\phi]$, respectively. In particular, we have $\left\langle \Psi_j^0, \Phi_\ell^0 \right\rangle = \delta_{j\ell}$ and the $\Phi_\ell^0$ and $\Psi_j^0$ satisfy the equations*

$$A_0[\phi]\Phi_1^0 = 0, \;\; A_0[\phi]\Phi_2^0 = -kc_M\Phi_1^0, \;\; A_0[\phi]\Phi_3^0 = -kc_Q\Phi_1^0$$

*and*

$$A_0^\dagger[\phi]\Psi_2^0 = 0 = A_0^\dagger[\phi]\Psi_3^0, \;\; A_0^\dagger[\phi]\Psi_1^0 \in \operatorname{span}\{\Psi_2^0, \Psi_3^0\} \setminus \{0\}.$$

Before continuing, we note for future use that the function $\phi_k$ is also 1-periodic and satisfies the equation

$$A[\phi]\phi_k = -kc_k\phi' - 2k^2cG^{-1}[\phi]\left((\phi')^2 - \phi\phi''\right)'. \tag{2.4.15}$$

Furthermore, differentiating (2.4.12) with respect to $k$, while holding $M$ and $Q$ constant, yields the important relations

$$\langle 1, \phi_k \rangle = 0 \quad \text{and} \quad \left\langle G^\dagger[\phi]\phi^{-2}, \phi_k \right\rangle = 2k \left\langle \phi^{-2}, (\phi')^2 \right\rangle. \tag{2.4.16}$$

### 2.4.3 Modulational Stability Calculation

Now that we have constructed a basis for the generalized kernels of $A_0[\phi]$ and its adjoint in a coordinate system compatible with the Whitham system (2.3.6), we proceed to study how this triple eigenvalue bifurcates from the $(\lambda, \xi) = (0, 0)$ state. To this end, recall from (1.3.6) that this is equivalent to seeking the 1-periodic eigenvalues in a neighborhood of the origin of the associated Bloch operators

$$A_\xi[\phi] = G_\xi^{-1}[\phi]k(\partial_\theta + i\xi)\mathcal{L}_\xi[\phi]$$

for $|\xi| \ll 1$, where here $G_\xi[\phi] := e^{-i\xi\theta}G[\phi]e^{i\xi\theta}$ and $\mathcal{L}_\xi[\phi] := e^{-i\xi\theta}\mathcal{L}[\phi]e^{i\xi\theta}$ are the Bloch operators associated with the operators $G[\phi]$ and $\mathcal{L}[\phi]$ defined in (2.4.13). Following [5, 38, 3], we begin expanding the Bloch operators for $|\xi| \ll 1$. As these expressions are analytic in $\xi$, it is straightforward to verify that

$$\mathcal{L}_\xi[\phi] = L_0 + (ik\xi)L_1 + (ik\xi)^2 L_2 \;\text{ and }\; G_\xi[\phi] = \widetilde{G}_0 + (ik\xi)\widetilde{G}_1 + (ik\xi)^2\widetilde{G}_2,$$

where here

$$L_0 := \mathcal{L}[\phi], \;\; L_1 := 2kc\left(\phi' - \phi\partial_\theta\right), \quad L_2 := -c\phi$$

and

$$\widetilde{G}_0 := G[\phi], \quad \widetilde{G}_1 := -2\phi k\partial_\theta, \quad \widetilde{G}_2 := -\phi.$$

are operators acting on $L^2_{\mathrm{per}}(0,1)$. Using that $\widetilde{G}_0[\phi]$ is invertible on $L^2_{\mathrm{per}}(0,1)$ and rewriting the expansion for $G_\xi[\phi]$ as

$$G_\xi[\phi] = \left(I + (ik\xi)\widetilde{G}_1\widetilde{G}_0^{-1} + (ik\xi)^2\widetilde{G}_2\widetilde{G}_0^{-1}\right)\widetilde{G}_0,$$

it follows $G_\xi^{-1}[\phi]$ can be expanded for $|\xi| \ll 1$ as the Neumann series

$$\begin{aligned} G_\xi^{-1}[\phi] &= \widetilde{G}_0^{-1}\sum_{\ell=0}^{\infty}(-1)^\ell[((ik\xi)\widetilde{G}_1 + (ik\xi)^2\widetilde{G}_2)\widetilde{G}_0^{-1}]^\ell \\ &= \mathcal{G}_0 + (ik\xi)\mathcal{G}_1 + (ik\xi)^2\mathcal{G}_2 + O(|\xi|^3), \end{aligned}$$

where here

$$\mathcal{G}_0 = G^{-1}[\phi], \quad \mathcal{G}_1 = 2G^{-1}[\phi](\phi k\partial_\theta(G^{-1}[\phi]\,\cdot))$$

and

$$\mathcal{G}_2 = G^{-1}[\phi](\phi G^{-1}[\phi]\,\cdot) + 4G^{-1}[\phi](\phi k\partial_\theta(G^{-1}[\phi](\phi k\partial_\theta(G^{-1}[\phi]\,\cdot))))$$

are again acting on $L^2_{\mathrm{per}}(0,1)$. The Bloch operators $A_\xi[\phi]$ can thus be expanded for $|\xi| \ll 1$ as

$$\begin{aligned} A_\xi[\phi] &= (\mathcal{G}_0 + (ik\xi)\mathcal{G}_1 + (ik\xi)^2\mathcal{G}_2 + O(|\xi|^3))(k\partial_\theta + ik\xi)(L_0 + (ik\xi)L_1 + (ik\xi)^2L_2) \\ &= A_0 + (ik\xi)A^{(1)} + (ik\xi)^2A^{(2)} + O(|\xi|^3) \end{aligned}$$

where, after some manipulation,

$$\begin{cases} A_0 = A_0[\phi], \\ A^{(1)} = \mathcal{G}_1 k\partial_\theta L_0 + \mathcal{G}_0 k\partial_\theta L_1 + \mathcal{G}_0 L_0 \\ \qquad = \mathcal{G}_0\left(2\phi k\partial_\theta A_0 + k\partial_\theta L_1 + L_0\right), \\ A^{(2)} = \mathcal{G}_2 k\partial_\theta L_0 + \mathcal{G}_1 k\partial_\theta L_1 + \mathcal{G}_0 k\partial_\theta L_2 + \mathcal{G}_1 L_0 + \mathcal{G}_0 L_1 \\ \qquad = \mathcal{G}_0\left(\phi A_0 + 4\phi k\partial_\theta \mathcal{G}_0 \phi k\partial_\theta A_0 + 2\phi k\partial_\theta \mathcal{G}_0 k\partial_\theta L_1 + k\partial_\theta L_2 + 2\phi k\partial_\theta \mathcal{G}_0 L_0 + L_1\right). \end{cases} \tag{2.4.17}$$

Note to assist in our computations of the actions of $A^{(1)}$ and $A^{(2)}$ later, we have expanded these operators in (2.4.17) as to identify any factors of $A_0$ present in them, as well as to pull out a global factor of $\mathcal{G}_0$.

Now, by Corollary 2.4.7, we know $\lambda = 0$ is an isolated eigenvalue of $A_0[\phi]$ with algebraic multiplicity three. Since $A_\xi[\phi]$ is a relatively compact perturbation of $A_0[\phi]$ for all $|\xi| \ll 1$ depending analytically on the Bloch parameter $\xi$, it follows that the operator $A_\xi[\phi]$ will have three eigenvalues $\{\lambda_j(\xi)\}_{j=1}^3$, defined for $|\xi| \ll 1$, bifurcating from $\lambda = 0$ for $0 < |\xi| \ll 1$. The modulational stability or instability of $\phi$ may then be determined by tracking these three eigenvalues for $|\xi| \ll 1$. To this end, observe we may use the bases identified in Corollary 2.4.7 to build explicit rank 3 spectral projections onto the generalized kernels of $A_0[\phi]$ and $A_0^\dagger[\phi]$. By standard spectral perturbation theory (see, for example, Theorems 1.7 and 1.8 in [41, Chapter VII.1.3]) these dual bases extend analytically into dual right and left bases $\{\Phi_\ell^\xi\}_{\ell=1}^3$ and $\{\Psi_j^\xi\}_{j=1}^3$ associated to the three eigenvalues $\{\lambda_j(\xi)\}_{j=1}^3$ near the origin that satisfy $\langle \Psi_j^\xi, \Phi_\ell^\xi \rangle = \delta_{j\ell}$ for all $|\xi| \ll 1$. For $|\xi| \ll 1$, we may now construct $\xi$-dependent rank 3 eigenprojections

$$\Pi(\xi) : L^2_{\text{per}}(0, 2\pi) \to \bigoplus_{j=1}^3 \ker\left(A_\xi[\phi] - \lambda_j(\xi) I\right),$$
$$\widetilde{\Pi}(\xi) : L^2_{\text{per}}(0, 2\pi) \to \bigoplus_{j=1}^3 \ker\left(A_\xi^\dagger[\phi] - \overline{\lambda_j(\xi)} I\right),$$

with ranges coinciding with the total left and right eigenspaces associated with the eigenvalues $\{\lambda_j(\xi)\}_{j=1}^3$. Using this one-parameter family of eigenprojections, for each fixed $|\xi| \ll 1$ we can project the infinite dimensional spectral problem for $A_\xi[\phi]$ onto the three-dimensional total eigenspace associated with the eigenvalues $\{\lambda_j(\xi)\}_{j=1}^3$. In particular, the action of the operators $A_\xi[\phi]$ on this subspace can be represented by the $3 \times 3$ matrix operator

$$D_\xi := \widetilde{\Pi}(\xi) A_\xi[\phi] \Pi(\xi) = \left(\left\langle \Psi_j^\xi, A_\xi[\phi] \Phi_\ell^\xi \right\rangle\right)_{j,\ell=1}^3.$$

It follows that for each $|\xi| \ll 1$ the eigenvalues $\lambda_j(\xi)$ correspond precisely to the values $\lambda$ where the matrix $D_\xi - \lambda I$ is singular, where here $I$ denotes the identity matrix[13] on $\mathbb{R}^3$. In what follows, we aim to explicitly construct this matrix for $|\xi| \ll 1$.

First, we observe that Corollary 2.4.7 implies that $D_\xi$ at $\xi = 0$ is given by[14]

$$D_0 = \left(\begin{array}{c|cc} 0 & -kc_M & -kc_Q \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Expanding the right and left bases $\Phi_\ell^\xi$ and $\Psi_j^\xi$ as[15]

$$\Phi_\ell^\xi = \Phi_\ell^0 + (ik\xi)\left(\frac{1}{ik}\partial_\xi \left.\Phi_\ell^\xi\right|_{\xi=0}\right) + (ik\xi)^2\left(\frac{1}{(ik)^2}\partial_\xi^2 \left.\Phi_\ell^\xi\right|_{\xi=0}\right) + O(|\xi|^3)$$

and

$$\Psi_j^\xi = \Psi_j^0 + (ik\xi)\left(\frac{1}{ik}\partial_\xi \left.\Psi_j^\xi\right|_{\xi=0}\right) + (ik\xi)^2\left(\frac{1}{(ik)^2}\partial_\xi^2 \left.\Psi_j^\xi\right|_{\xi=0}\right) + O(|\xi|^3),$$

yields an expansion of the matrix $D_\xi$ of the form

$$D_\xi = D_0 + (ik\xi)D^{(1)} + (ik\xi)^2 D^{(2)} + O(|\xi|^3).$$

Note that, explicitly,

$$D^{(1)} = \left(\left\langle \Psi_j^0, A_0\frac{1}{ik}\partial_\xi \left.\Phi_\ell^\xi\right|_{\xi=0} + A^{(1)}\Phi_\ell^0\right\rangle + \left\langle \frac{1}{ik}\partial_\xi \left.\Psi_j^\xi\right|_{\xi=0}, A_0\Phi_\ell^0\right\rangle\right)_{j,\ell=1}^3.$$

To continue, we need information regarding $\partial_\xi\Phi_1^\xi$ at $\xi = 0$. We claim that, up to harmless modifications of the basis functions used above, we can arrange for this first order variation to be exactly $\phi_k$, while simultaneously preserving biorthogonality of the bases up to $O(|\xi|^2)$. Indeed, observe that by differentiating the identity $\Pi(\xi)A_\xi[\phi]\Phi_1^\xi = A_\xi[\phi]\Phi_1^\xi$ and evaluating at $\xi = 0$ yields

[13]Here, we are using that $\widetilde{\Pi}(\xi)\Pi(\xi)$ is the identity by construction.

[14]The horizontal and vertical lines are included only to organize the $3 \times 3$ matrices into sub-blocks.

[15]Note the expansion in $ik\xi$, rather than simply $i\xi$, is natural due to our spatial rescaling. Further, it is consistent with how $A_\xi[\phi]$ naturally expands in the rescaled system.

the relation

$$\Pi(0)\left(A_0\left(\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}\right)+A^{(1)}\Phi_1^0\right)+\frac{1}{ik}\partial_\xi\ \Pi(\xi)|_{\xi=0}\,A_0\Phi_1^0=A_0\left(\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}\right)+A^{(1)}\Phi_1^0.$$

Recalling $A_0\Phi_1^0=0$, and noting that (2.4.15) can be rewritten as $A^{(1)}\Phi_1^0=-A_0\phi_k-kc_k\Phi_1^0$, we find that

$$\Pi(0)\left(A_0\left(\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}-\phi_k\right)\right)=A_0\left(\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}-\phi_k\right)$$

which implies that $\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}-\phi_k$ lies in the generalized kernel of $A_0[\phi]$. Consequently, there exists constants $\{a_j\}_{j=1}^3$ such that

$$\frac{1}{ik}\partial_\xi\ \Phi_1^\xi\Big|_{\xi=0}=\phi_k+\sum_{j=1}^3 a_j\Phi_j^0.$$

Replacing $\Phi_1^\xi$ with

$$\widetilde{\Phi}_1^\xi:=\Phi_1^\xi-(ik\xi)\sum_{j=1}^3 a_j\Phi_j^0$$

while simultaneously replacing $\Psi_j^\xi$ for $j=1,2,3$ with

$$\widetilde{\Psi}_j^\xi:=\Psi_j^\xi+(ik\xi)a_j\Psi_1^\xi,$$

we readily see that

$$\widetilde{\Phi}_1^\xi=\Phi_1^0+(ik\xi)\phi_k+O(|\xi|^2),\quad I_\xi=\widetilde{\Pi}(\xi)\Pi(\xi)=I+O(|\xi|^2), \tag{2.4.18}$$

as claimed. Note that $I_\xi$ is the $3\times 3$ matrix describing the action of the identity operator with respect to the modified bases. For notational simplicity, we will drop the tildes throughout the remainder and refer to these modified bases as simply $\{\Phi_\ell^\xi\}_{\ell=1}^3$ and $\{\Psi_j^\xi\}_{j=1}^3$ With the above choices, the terms involving the variations in $\Psi_j^\xi$ can be directly computed. For example, using Corollary 2.4.7 we have

$$\left\langle \frac{1}{ik}\partial_\xi \left.\Psi_j^\xi\right|_{\xi=0}, A_0\Phi_2^0 \right\rangle = -kc_M \left\langle \frac{1}{ik}\partial_\xi \left.\Psi_j^\xi\right|_{\xi=0}, \Phi_1^0 \right\rangle = kc_M \left\langle \Psi_j^\xi, \frac{1}{ik}\partial_\xi \left.\Phi_1^0\right|_{\xi=0} \right\rangle = kc_M \left\langle \Psi_j^0, \phi_k \right\rangle,$$

where the second equality follows since (2.4.18)(ii) implies

$$0 = \partial_\xi \left.\left\langle \Psi_j^\xi, \Phi_1^\xi \right\rangle\right|_{\xi=0} = \left\langle \partial_\xi \left.\Psi_j^\xi\right|_{\xi=0}, \Phi_1^0 \right\rangle + \left\langle \Psi_j^\xi, \partial_\xi \left.\Phi_1^0\right|_{\xi=0} \right\rangle.$$

Using the above modified bases, straight forward calculations yield

$$D^{(1)} = \left(\begin{array}{c|cc} -kc_k & * & * \\ \hline 0 & \left\langle \Psi_2^0, A^{(1)}\Phi_2^0 \right\rangle & \left\langle \Psi_2^0, A^{(1)}\Phi_3^0 \right\rangle \\ 0 & \left\langle \Psi_3^0, A^{(1)}\Phi_2^0 + kc_M\phi_k \right\rangle & \left\langle \Psi_3^0, A^{(1)}\Phi_3^0 + kc_Q\phi_k \right\rangle \end{array}\right),$$

where here "$*$" is used to denote undetermined terms that, as we will see, are irrelevant to our calculation. Similarly, one finds

$$D^{(2)} = \left(\begin{array}{c|cc} * & * & * \\ \hline \left\langle \Psi_2^0, A^{(2)}\Phi_1^0 + A^{(1)}\phi_k \right\rangle & * & * \\ \left\langle \Psi_3^0, A^{(2)}\Phi_1^0 + A^{(1)}\phi_k + kc_k\phi_k \right\rangle & * & * \end{array}\right)$$

Noticing the above implies the matrix $D_\xi$ satisfies

$$D_\xi = \left(\begin{array}{c|c} O(|\xi|) & O(1) \\ \hline O(|\xi|^2) & O(|\xi|) \end{array}\right),$$

where the upper left block is $1 \times 1$, it follows by standard arguments that the eigenvalues $\lambda_j(\xi)$ are at least $C^1$ in $\xi$, and can thus be written as $\lambda_j(\xi) = ik\xi\mu_j(\xi)$ for some continuous functions $\mu_j$

defined for $|\xi| \ll 1$. Further, introducing, for $0 < |\xi| \ll 1$, the invertible matrix[16]

$$S(\xi) := \left(\begin{array}{c|cc} ik\xi & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

and defining $\widehat{D}_\xi := \frac{1}{ik\xi} S(\xi) D_\xi S(\xi)^{-1}$ and $\widehat{I}_\xi := S(\xi) I_\xi S(\xi)^{-1}$, it follows $\widehat{D}_\xi$ and $\widehat{I}_\xi$ are analytic in $ik\xi$ and, at $\xi = 0$, are given by $\widehat{I}_0 = I$ and

$$\widehat{D}_0 = \begin{pmatrix} -kc_k & -kc_M & -kc_Q \\ \left\langle \Psi_2^0, A^{(2)}\Phi_1^0 + A^{(1)}\phi_k \right\rangle & \left\langle \Psi_2^0, A^{(1)}\Phi_2^0 \right\rangle & \left\langle \Psi_2^0, A^{(1)}\Phi_3^0 \right\rangle \\ \left\langle \Psi_3^0, A^{(2)}\Phi_1^0 + A^{(1)}\phi_k + kc_k\phi_k \right\rangle & \left\langle \Psi_3^0, A^{(1)}\Phi_2^0 + kc_M\phi_k \right\rangle & \left\langle \Psi_3^0, A^{(1)}\Phi_3^0 + kc_Q\phi_k \right\rangle \end{pmatrix} \tag{2.4.19}$$

Furthermore, the $\mu_j(\xi)$ are the eigenvalues of the matrix $\widehat{D}_\xi$ since clearly

$$\det(D_\xi - \lambda(\xi) I_\xi) = (ik\xi)^3 \det(\widehat{D}_\xi - \mu(\xi)\widehat{I}_\xi)$$

In summary, we have proven the following result.

**Theorem 2.4.8.** *Under the hypotheses of Corollary 2.4.7, in a sufficiently small neighborhood of the origin, the spectrum of $A[\phi]$ on $L^2(\mathbb{R})$ consists of precisely three $C^1$ curves $\{\lambda_j(\xi)\}_{j=1}^3$ defined for $|\xi| \ll 1$ which can be expanded as*

$$\lambda_j(\xi) = ik\xi\mu_j(0) + o(|\xi|), \quad j = 1, 2, 3$$

*where the $\mu_j(0)$ are precisely the eigenvalues of the matrix $\widehat{D}_0$ above. In particular, a necessary condition for $\phi$ to be a spectrally stable solution of* (2.1.1) *is that all the eigenvalues of $\widehat{D}_0$ are real.*

Note the possible spectral instability predicted from Theorem 2.4.8 is of modulational type, oc-

[16]Note that the conjugating by $S(\xi)$ effectively replaces the coefficient of the $\phi'$, corresponding to the local phase $\psi$, with the wave number $|\xi|\psi \sim \psi_x$. This is reminiscent of the fact that the Whitham system (2.3.6) involves the wave number $\psi_x$, rather than the phase $\psi$.

curring near the origin in the spectral plane for side-band Bloch frequencies. In general, computing the eigenvalues of $\widehat{D}_0$ is a difficult task, requiring one to identify the above inner products in terms of known quantities: see, for example, [7, 5]. As we will see, however, this identification is not necessary in order to rigorously connect modulational instability of $\phi$ to the Whitham modulation system (2.3.6). Indeed, recalling the quasilinear form (2.3.7) of the Whitham modulation system (2.3.6), in the next section we will prove that

$$\mathbf{D}(\phi) = \widehat{D}_0 - cI$$

so that, in particular, a necessary condition for the spectral stability of $\phi$ is that the matrix $\mathbf{D}(\phi)$ is weakly hyperbolic, i.e. that all of its eigenvalues are real, thus establishing Theorem 2.3.1.

## 2.5 Proof of Theorem 2.3.1

In this section, we establish Theorem 2.3.1. We will use a direct, row-by-row calculation to show that

$$\mathbf{D}(\phi) = \widehat{D}_0 - cI,$$

where here $\mathbf{D}(\phi)$ is the linearized matrix associated to the Whitham modulation equations (2.3.7) and the eigenvalues of $\widehat{D}_0$, defined in (2.4.19), rigorously describe the structure of the $L^2(\mathbb{R})$-spectrum of the linearized operator $A[\phi]$ in a sufficiently small neighborhood of the origin: see Theorem 2.4.8. To this end, first observe that the first rows of each of these matrices are clearly identical. To compare the second rows, it is enough to establish the identities

$$\begin{cases} \left\langle 1, A^{(2)}\phi' + A^{(1)}\phi_k \right\rangle = \left\langle 1, 2k^2 c(\phi')^2 - \phi^2 \right\rangle_k, \\ \left\langle 1, A^{(1)}\phi_M \right\rangle - c = \left\langle 1, 2k^2 c(\phi')^2 - \phi^2 \right\rangle_M, \\ \left\langle 1, A^{(1)}\phi_Q \right\rangle = \left\langle 1, 2k^2 c(\phi')^2 - \phi^2 \right\rangle_Q. \end{cases} \tag{2.5.1}$$

For the first equality, observe from (2.4.17) that, after some manipulations,

$$A^{(2)}\phi' + A^{(1)}\phi_k = G^{-1}[\phi]\left(kc(\phi')^2 - 3kc\phi\phi'' - 2k^2c_k\phi\phi'' + 2k^2c\partial_\theta(\phi'\phi_k - \phi\phi_k') + \mathcal{L}[\phi]\phi_k\right) \tag{2.5.2}$$

which, recalling that $G^{-\dagger}[\phi](1) = 1$ and using integration by parts, yields

$$\left\langle 1, A^{(2)}\phi' + A^{(1)}\phi_k \right\rangle = \left\langle 1, 4kc(\phi')^2 + 2k^2c_k(\phi')^2 + \mathcal{L}[\phi]\phi_k \right\rangle.$$

Since integration by parts implies

$$\begin{aligned}\langle 1, \mathcal{L}[\phi]\phi_k\rangle &= \left\langle 1, c\phi_k - 2\phi\phi_k - k^2c\phi''\phi_k + 2k^2c\phi'\phi_k' - k^2c\phi\phi_k''\right\rangle \\ &= c\,\langle 1, \phi_k\rangle + \left\langle 1, -(\phi^2)_k + 2k^2c((\phi')^2)_k\right\rangle\end{aligned}$$

it follows that

$$\left\langle 1, A^{(2)}\phi' + A^{(1)}\phi_k \right\rangle = c\,\langle 1, \phi_k\rangle + \left\langle 1, 2k^2c(\phi')^2 - \phi^2\right\rangle_k,$$

which, recalling $\langle 1, \phi_k\rangle = 0$ by (2.4.16)(i), establishes (2.5.1)(i). The other two equalities in (2.5.1) follow in a similar way. Indeed, using analogous manipulations as above, as well as integration by parts, we find

$$\left\langle 1, A^{(1)}\phi_M \right\rangle = c\,\langle 1, \phi_M\rangle + \left\langle 1, 2k^2c(\phi')^2 - \phi^2\right\rangle_M$$

and

$$\left\langle 1, A^{(1)}\phi_Q \right\rangle = c\,\left\langle 1, \phi_Q\right\rangle + \left\langle 1, 2k^2c(\phi')^2 - \phi^2\right\rangle_Q.$$

Since $\langle 1, \phi_M\rangle = 1$ and $\left\langle 1, \phi_Q\right\rangle = 0$ by the biorthogonality relations in Corollary (2.4.7), the identities (2.5.1)(ii)-(iii) follow. This establishes that the second rows of the matrices $\mathbf{D}(\phi)$ and $\widehat{D}_0 - cI$ are identical.

To compare the third rows, we aim to establish the following three identities:

$$\begin{cases} \left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(2)}\phi' + A^{(1)}\phi_k + kc_k\phi_k \right\rangle = -\langle 1, 2\ln|\phi|\rangle_k\,, \\ \left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(1)}\phi_M + kc_M\phi_k \right\rangle = -\langle 1, 2\ln|\phi|\rangle_M \\ \left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(1)}\phi_Q + kc_Q\phi_k \right\rangle + c = -\langle 1, 2\ln|\phi|\rangle_Q \end{cases} \tag{2.5.3}$$

Focusing on the first term, we note that (2.5.2) and integration by parts implies

$$\begin{aligned} \left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(2)}\phi' + A^{(1)}\phi_k + kc_k\phi_k \right\rangle &= \left\langle \phi^{-2}, -2k(c+kc_k)(\phi')^2 + 2k^2c(\phi''\phi_k - \phi\phi_k'') + \mathcal{L}[\phi]\phi_k \right\rangle \\ &\quad + \left\langle G^{\dagger}[\phi]\phi^{-2}, kc_k\phi_k \right\rangle \end{aligned}$$

so that, by (2.4.16)(ii),

$$\left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(2)}\phi' + A^{(1)}\phi_k + kc_k\phi_k \right\rangle = -c\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_k \right\rangle + \left\langle \phi^{-2}, 2k^2c(\phi''\phi_k - \phi\phi_k'') + \mathcal{L}[\phi]\phi_k \right\rangle.$$

Now, using the fact that $\left\langle \phi^{-2}, \phi\phi_k'' \right\rangle = \left\langle \phi^{-2}, \phi'\phi_k' \right\rangle$ we see

$$\begin{aligned} \left\langle \phi^{-2}, \mathcal{L}[\phi]\phi_k \right\rangle &= -\left\langle \phi^{-2}, 2\phi\phi_k \right\rangle + \left\langle \phi^{-2}, c\phi_k - k^2c\phi''\phi_k + 2k^2c\phi'\phi_k' - k^2c\phi\phi_k'' \right\rangle \\ &= -2\langle 1, \ln|\phi|\rangle_k + \left\langle \phi^{-2}, c\phi_k - k^2c\left(\phi''\phi_k - \phi\phi_k''\right) \right\rangle \end{aligned}$$

and hence

$$\begin{aligned} \left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(2)}\phi' + A^{(1)}\phi_k + kc_k\phi_k \right\rangle &= -c\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_k \right\rangle - \langle 1, 2\ln|\phi|\rangle_k \\ &\quad + \left\langle \phi^{-2}, c\phi_k + k^2c\left(\phi''\phi_k - \phi\phi_k''\right) \right\rangle \\ &= -c\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_k \right\rangle - \langle 1, 2\ln|\phi|\rangle_k + c\left\langle \phi^{-2}, G[\phi]\phi_k \right\rangle. \end{aligned}$$

The identity (2.5.3)(i) has now been established. The other two equalities follow similarly. Indeed, using analogous manipulations as above, as well as integration by parts, we find that

$$\left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(1)}\phi_M + kc_M\phi_k \right\rangle = -\langle 1, 2\ln|\phi| \rangle_M + c\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_M \right\rangle$$

and

$$\left\langle G^{\dagger}[\phi]\phi^{-2}, A^{(1)}\phi_Q + kc_Q\phi_k \right\rangle = -\langle 1, 2\ln|\phi| \rangle_Q + c\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_Q \right\rangle.$$

Recalling that $\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_M \right\rangle = 0$ and $\left\langle G^{\dagger}[\phi]\phi^{-2}, \phi_Q \right\rangle = -1$ by the biorthogonality relations in Corollary (2.4.7), the identities (2.4.16)(ii)-(iii) follow. This complete the proof of Theorem 2.3.1

## 2.6 Analysis for Small Amplitude Waves

In general, determining whether the Whitham system (2.3.6) is weakly hyperbolic at a given periodic traveling solution of (2.1.1) is a difficult matter. See [5, 7], for example, for cases where this information can be computed in the presence of a Hamiltonian structure. Nevertheless, one can use well-conditioned numerical methods to approximate the entries of the matrix $\mathbf{D}(\phi)$ in (2.3.7), thereby producing a numerical stability diagram. Such numerical analysis was recently carried out in [47]. There, the authors findings indicate, among other things, that the Whitham modulation system (2.3.6) is hyperbolic about a given periodic traveling wave $\phi$ provided $\phi$ has sufficiently large period, while the system is elliptic for sufficiently small periods. Formally then, the authors findings suggest long waves are modulationally stable while short waves are modulationally unstable. For asymptotically small waves, however, it is possible to use asymptotic analysis to analyze the hyperbolicity of the Whitham modulation system (2.3.6). While this analysis is discussed in [47], for completeness we reproduce their result for small amplitude waves.

To this end, we note that $T = 1/k$-periodic traveling wave solutions with speed $c$ of the conduit equation (2.1.1) correspond to 1-periodic stationary solutions of the profile equation

$$-\omega\phi' + 2k\phi\phi' + k^2\omega\phi\phi''' - k^2\omega\phi'\phi'' = 0, \tag{2.6.1}$$

where where $\omega = kc$ is the frequency and primes denote differentiation with respect to the traveling variable $\theta = kx - \omega t$. Using an elementary Lyapunov-Schmidt argument, one can show that solution pairs $(\phi, \omega)$ of (2.6.1) with asymptotically small oscillations about its mean[17] $M$ admit a convergent asymptotic expansion in $H^3_{\text{per}}(0,1)$ of the form

$$\phi(\theta; k, M, A) = M + A\cos(2\pi\theta) + \sum_{j=2}^{\infty} A^j \phi_j(\theta; k, M)$$
$$\omega(k, M, A) = \omega_0(k, M) + A^2\omega_2(k, M) + O(A^4)$$

valid for $|A| \ll 1$, where the functions $\phi_j$ are 1-periodic and satisfy

$$\int_0^1 \phi_j(\theta)d\theta = 0 = \int_0^1 \phi_j(\theta)\cos(2\pi\theta)d\theta$$

for all $j \geq 2$. Further, it is an easy calculation to see that

$$\phi_2(\theta) = \frac{1}{6(2\pi)^2 k\omega_0 M}\cos(4\pi\theta) = \frac{(2\pi)^2k^2M + 1}{12(2\pi)^2k^2M^2}\cos(4\pi\theta)$$

and

$$\omega_0 = \frac{2kM}{(2\pi)^2k^2M + 1}, \quad \omega_2 = \frac{1 - 8(2\pi)^2k^2M}{12(2\pi)^2kM^2((2\pi)^2k^2M + 1)}.$$

Using these asymptotic expansions, the Whitham modulation system (2.3.6) about these waves expands for $|A| \ll 1$ as

$$\begin{cases} k_S + \partial_X\left(\omega_0 + A^2\omega_2\right) = O(A^3) \\ M_S + \partial_X\left(M^2 - \frac{1}{2}A^2\left(2(2\pi)^2k\omega_0 - 1\right)\right) = O(A^3) \\ \left(1 + O(A^2)\right)\left(A^2\right)_S + A^2\frac{\partial^2\omega_0}{\partial k^2}k_X + \left(A^2\right)_X\frac{\partial\omega_0}{\partial k} + 4A^2\left(1 - \frac{2(2\pi)^2k^2M}{(2(2\pi)^2k^2M + 1)^3}\right)M_X = O(A^3). \end{cases}$$

[17]Note that, here, the mean $M$ agrees with the mass, as defined in (2.2.2), since we are working with 1-periodic functions. More generally, these quantities would differ by factor of $k$.

Using the chain rule, this system can be rewritten in the quasilinear form

$$\begin{pmatrix} k \\ M \\ A \end{pmatrix}_S + B(k, M, A) \begin{pmatrix} k \\ M \\ A \end{pmatrix}_X = 0,$$

where here $B$ expands in $A$ as

$$B(k, M, A) = B_0(k, M) + AB_1(k, M) + A^2 \widetilde{B}(k, M, A),$$

with $\widetilde{B}(k, M, A)$ a bounded, continuous matrix-valued function and

$$B_0 = \begin{pmatrix} \frac{\partial \omega_0}{\partial k} & \frac{\partial \omega_0}{\partial M} & 0 \\ 0 & 2M & 0 \\ 0 & 0 & \frac{\partial \omega_0}{\partial k} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 2\omega_2 \\ 0 & 0 & 1 - 2(2\pi)^2 k\omega_0 \\ \frac{1}{2} \frac{\partial^2 \omega_0}{\partial k^2} & 2\left(1 - \frac{2(2\pi)^2 k^2 M}{(2(2\pi)^2 k^2 M + 1)^3}\right) & 0 \end{pmatrix}.$$

Based on the above asymptotics, the eigenvalues of the Whitham system (2.3.6) about the small amplitude 1-periodic traveling wave $\phi(\cdot; k, M, A)$ may be asymptotically expanded as

$$\begin{cases} \lambda_1(k, M, A) = 2M + \mathcal{O}(A^2) \\ \lambda_\pm(k, M, A) = \dfrac{\partial \omega_0}{\partial k} \pm A\sqrt{-n(k, M)\dfrac{\partial^2 \omega_0}{\partial k^2}} + \mathcal{O}(A^2), \end{cases}$$

where we have explicitly

$$n(k, M) = \frac{8[(2\pi)^2 k^2 M]^2 + 5[(2\pi)^2 k^2 M] + 3}{12(2\pi)^2 k M^2([(2\pi)^2 k^2 M] + 1)([(2\pi)^2 k^2 M] + 3)}.$$

Since

$$\frac{\partial^2 \omega_0}{\partial k^2}(k, M) = \frac{16\pi^2 k M^2 \left((2\pi k)^2 M - 3\right)}{((2\pi k))^2 M + 1)^3},$$

and since $n(k, M)$ is clearly a strictly positive function of $k$ and $M$, we have the following result.

**Theorem 2.6.1.** *Let $\phi(\cdot; k, M, A)$ be a* 1*-periodic traveling wave solution of* (2.6.1) *with asymptotically small amplitude. Then $\phi$ is modulationally unstable if*

$$(2\pi k)^2 > \frac{3}{M}.$$

*Further, a necessary condition for $\phi$ to be modulationally stable is*

$$0 < (2\pi k)^2 < \frac{3}{M}.$$

Theorem 2.6.1 makes rigorous the formal calcluations in [47] regarding small amplitude periodic traveling wave solutions[18] of (2.1.1). Note that while the above analysis shows the Whitham system (2.3.6) is (strictly) hyperbolic when $0 < (2\pi k)^2 < 3/M$, this is not sufficient to conclude modulational stability of the underlying wave $\phi$ since hyperbolicity of (2.3.6) only guarantees the eigenvalues of $\mathcal{A}_\xi[\phi]$ lie on the imaginary axis to first order in $\xi$, i.e. it guarantees tangency of the spectral curves at $\lambda = 0$ to the imaginary axis. Of course, modulational stability requires that the spectral curves near the origin are confined to the left half plane, and hence cannot be concluded[19] from only first order information. Modulational stability was concluded in [47] for the conduit equation, however, in the case $0 < (2\pi k)^2 < 3/M$ through numerical time evolution. It would be interesting to rigorously verify this prediction.

**Remark 2.6.2.** *Note that a slightly different approach to proving Theorem 2.3.1 would have been to expand the matrix $D(\phi)$ in* (2.3.7)*, and using the chain rule to express derivatives with respect to $Q$ in terms of derivatives with respect to $(k, M, A)$. This calculation of course produces the same result. However, here we preferred to start working directly with the variables $(k, M, A)$ in the Whitham system, since this is the natural parameterization in the asymptoically small amplitude limit.*

[18] In [47], however, note that $k$ was the wave number relative to $2\pi$-perturbations, while in our work $k$ denotes the wave number relative to 1-periodic perturbations. This accounts for the extra factor of $2\pi$ present in our result compared to that in [47].

[19] An exception to this rule occurs when the PDE admits a Hamiltonian strucutre. See, for example, related work on the KdV equation [6, 5].

# Chapter 3

# Linear Modulational and Subharmonic Dynamics of Spectrally Stable Periodic Waves in the Lugiato-Lefever Equation


## Abstract

In this chapter, we study the linear dynamics of spectrally stable $T$-periodic stationary solutions of the Lugiato-Lefever equation (LLE), a damped nonlinear Schrödinger equation with forcing that arises in nonlinear optics. Such $T$-periodic solutions are nonlinearly stable to $NT$-periodic, i.e. subharmonic, perturbations for each $N \in \mathbb{N}$ with exponential decay rates of perturbations of the form $e^{-\delta_N t}$. However, both the exponential rates of decay $\delta_N$ and the allowable size of the initial perturbations tend to 0 as $N \to \infty$, so that this result is non-uniform in $N$ and, in fact, empty in the limit $N = \infty$. The primary goal of this chapter is to introduce a methodology, in the context of the LLE, by which a uniform stability result for subharmonic perturbations may be achieved, at least at the linear level. The obtained uniform decay rates are shown to agree precisely with the polynomial decay rates of localized, i.e. integrable on the real line, perturbations of such spectrally stable periodic solutions of the LLE. This work both unifies and expands on several existing works in the literature concerning the stability and dynamics of such waves, and sets forth a general methodology for studying such problems in other contexts. This chapter is representative of [21], which was a joint work with Mariana Haragus and Mathew Johnson.

## 3.1 Introduction

In this chapter, we consider the stability and dynamics of periodic stationary solutions of the Lugiato-Lefever equation (LLE)

$$\psi_t = -i\beta\psi_{xx} - (1 + i\alpha)\psi + i|\psi|^2\psi + F, \tag{3.1.1}$$

where $\psi(x,t)$ is a complex-valued function depending on a temporal variable $t$ and a spatial variable $x$, the parameters $\alpha, \beta$ are real, and $F$ is a positive constant. The model (3.1.1) was derived from Maxwell's equations in [45] as a model to study pattern formation within the optical field in a dissipative and nonlinear optical cavity filled with a Kerr medium and subjected to a continuous laser pump. In this context, $\psi(x,t)$ represents the field envelope, $\alpha > 0$ represents a detuning parameter, $F > 0$ represents a normalized pump strength, and $|\beta| = 1$ is the dispersion parameter. The case $\beta = 1$ is referred to as the "normal" dispersion case while $\beta = -1$ is referred to as the "anomalous" dispersion case.

Since its derivation, the LLE has been intensely studied in the physics literature in the context of nonlinear optics, having more recently become a model for high-frequency optical combs generated by microresonators in periodic optical wave guides (see, for example, [9] and references therein). Until recently, however, there have been relatively few mathematically rigorous studies of the LLE. Several recent works have established the existence of periodic standing solutions of (3.1.1). Such solutions $\psi(x,t) = \phi(x)$ correspond to $T$-periodic solutions of the profile equation

$$-i\beta\phi'' - (1 + i\alpha)\phi + i|\phi|^2\phi + F = 0. \tag{3.1.2}$$

Using tools from bifurcation theory, the existence of periodic standing waves bifurcating both locally and globally from constant solutions has been shown in [51, 49, 12, 11]. Another type of periodic solutions has been recently constructed in the case of anomalous dispersion $\beta = -1$ in [19]. These solutions correspond to bifurcations from the standard arbitrary amplitude dnoidal solutions of the cubic NLS equation, for small $|(F,\alpha)|$. We also refer to [18] for a local bifurcation analysis of bounded solutions of (3.1.2) including, besides periodic, also localized and quasi-periodic solutions.

Our work focuses on the dynamical stability and long-time asymptotic dynamics of spatially periodic standing solutions of (3.1.1) when subject to varying classes of perturbations. Note that if $\phi$ is a $T$-periodic standing solution of (3.1.1) and we decompose $\phi = \phi_r + i\phi_i$ into its real and imaginary parts, then a function of the form $\psi(x,t) = \phi(x) + v(x,t)$, with $v = v_r + iv_i$, is a solution of (3.1.1) provided it satisfies the real system

$$\partial_t \begin{pmatrix} v_r \\ v_i \end{pmatrix} = \mathcal{A}[\phi] \begin{pmatrix} v_r \\ v_i \end{pmatrix} + \mathcal{N}(v), \tag{3.1.3}$$

where here $\mathcal{N}(v)$ is at least quadratic in $v$ and $\mathcal{A}[\phi]$ is the (real) linear differential operator

$$\mathcal{A}[\phi] = -I + \mathcal{J}\mathcal{L}[\phi], \tag{3.1.4}$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}[\phi] = \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}.$$

The choice of a function space for the evolution problem (3.1.3) is determined by the class of perturbations of the $T$-periodic standing wave. Choosing a Hilbertian framework, we take $L^2_{\text{per}}(0,T)$ for co-periodic perturbations, $L^2_{\text{per}}(0,NT)$ with $N \in \mathbb{N}$ for so-called subharmonic perturbations, and $L^2(\mathbb{R})$ for localized perturbations.[1] Recall that the spectral stability of a periodic wave $\phi$ to a given class of perturbations is determined by the spectrum of the linear operator $\mathcal{A}[\phi]$ when acting on the associated function space. Similarly, the linear stability of $\phi$ is given by the properties of the associated evolution semigroup $(e^{\mathcal{A}[\phi]t})_{t\geq 0}$, and the nonlinear (orbital) stability by the behavior of

[1]Since operators here are defined on vector valued functions, throughout this work we will abuse notation slightly and write $L^2(\mathbb{R})$ rather than $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and similarly for all other Lebesgue and Sobolev spaces. Furthermore, when the meaning is clear from context, we will write functions in $(f_1, f_2) \in L^2 \times L^2$ as simply $f \in L^2$.

the solutions of the nonlinear equation (3.1.3).

The spectral stability of periodic waves bifurcating locally from constant solutions has been studied in [11, 12]. It turns out that most of these waves are unstable for subharmonic perturbations, with $N$ larger than a certain value $N_c \geq 1$ depending on the parameters $\alpha$ and $F$, and that there is precisely one family of such waves which are spectrally stable for all subharmonic perturbations, and also for localized perturbations. These stable periodic waves bifurcate supercritically in the case of anomalous dispersion, $\beta = -1$, for any fixed parameter $\alpha < 41/30$ and bifurcation parameter $F^2 = F_1^2 + \mu$, for sufficiently small $\mu > 0$, where $F_1^2 = (1-\alpha)^2 + 1$. More precisely, there exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$, the LLE has an even periodic solution with Taylor expansion

$$\phi_\mu(x) = \phi^* + \frac{3(\alpha + i(2-\alpha))}{F_1(41 - 30\alpha)^{1/2}} \cos(\sqrt{2-\alpha}\, x)\, \mu^{1/2} + O(\mu), \tag{3.1.5}$$

where $\phi^*$ is the unique constant solution satisfying the algebraic equation

$$(1 + i\alpha)\phi - i\phi|\phi|^2 = F_1.$$

The solution $\phi_\mu$ is $T$-periodic with period $T = 2\pi/\sqrt{2-\alpha}$, and it is of class $C^\infty$. We point out that this parameter regime has been investigated in the original work of Lugiato and Lefever [45] who determined the value $\alpha_c = 41/30$ as an instability threshold.

For co-periodic perturbations which are $H^2$, i.e., belong to the domain of the linear operator $\mathcal{A}[\phi]$, the nonlinear asymptotic stability of the periodic waves (3.1.5) is a direct consequence of the bifurcation analysis used for their construction [51, 12]. Using Strichartz-type estimates, this result has been extended to more general $L^2_{\text{per}}(0,T)$-perturbations in [52]. As pointed out in [12, Section 6(a)], the bifurcation analysis used to construct these periodic waves can be extended to spaces of $NT$-periodic functions, for any arbitrary but fixed $N$, which then also gives a nonlinear stability result for these waves for $H^2$-subharmonic perturbations. However, this stability result is not uniform in $N$: for a given periodic wave $\phi_\mu$ as in (3.1.5), nonlinear stability is obtained for a finite number of integers $N$, only.[2]

For localized perturbations, as well as for general bounded perturbations, the spectral stability of the periodic waves (3.1.5) has been proved in [12, Theorem 4.3]. Based on Floquet-Bloch theory and spectral perturbation theory, this result shows that the periodic waves given by (3.1.5) are *diffusively spectrally stable* in the sense of Definition 1.3.1, which we repeat here for convenience.

**Definition 3.1.1.** A $T$-periodic stationary solution $\phi \in H^1_{\text{loc}}(\mathbb{R})$ of (3.1.1) is said to be *diffusively spectrally stable* provided the following conditions hold:

(i) the spectrum of the linear operator $\mathcal{A}[\phi]$ given by (3.1.4) and acting in $L^2(\mathbb{R})$ satisfies

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}[\phi]) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\} \cup \{0\};$$

(ii) there exists $\theta > 0$ such that for any $\xi \in [-\pi/T, \pi/T)$ the real part of the spectrum of the Bloch operator $\mathcal{A}_\xi[\phi] := e^{-i\xi x}\mathcal{A}[\phi]e^{i\xi x}$ acting on $L^2_{\text{per}}(0,T)$ satisfies

$$\Re\left(\sigma_{L^2_{\text{per}}(0,T)}(\mathcal{A}_\xi[\phi])\right) \leq -\theta\xi^2;$$

(iii) $\lambda = 0$ is a simple eigenvalue of $\mathcal{A}_0[\phi]$ with associated eigenvector the derivative $\phi'$ of the periodic wave.[3]

This stability notion was first introduced in [31] for more general classes of viscous conservation and balance laws, and is stated here in the context of (3.1.1). The properties (i)-(iii) in this definition are the main assumptions required for the present analysis. Recall from Section 1.3.1 that Floquet-Bloch theory shows that the spectrum of $\mathcal{A}[\phi]$ acting on $L^2(\mathbb{R})$ is equal to the union of the spectra of the Bloch operators $\mathcal{A}_\xi[\phi]$ acting on $L^2_{\text{per}}(0,T)$ for $\xi \in [-\pi/T, \pi/T)$. Similarly, in Section 1.3.4 we showed that, for subharmonic perturbations, the spectrum of the operator $\mathcal{A}[\phi]$ acting on $L^2_{\text{per}}(0,NT)$ is the union of the spectra of the Bloch operators $\mathcal{A}_\xi[\phi]$ acting on $L^2_{\text{per}}(0,T)$ for $\xi \in \Omega_N$, which is defined in (1.3.10) as a discrete subset of the interval $[-\pi/T, \pi/T)$ such that

[2]This is a result of their construction, which is based on center manifold techniques.

[3]Recall, by our slight abuse of notation above, here we are writing $\phi'$ instead of the technically correct $(\phi_r', \phi_i')$.

$e^{i\xi NT} = 1$. Recall from Remark 1.3.3 that this implies that diffusively spectrally stable periodic waves are spectrally stable for all subharmonic perturbations. In particular, we concluded that for such perturbations, the spectrum of $\mathcal{A}[\phi]$ is purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities, $\lambda = 0$ is a simple eigenvalue, with associated eigenvector the derivative $\phi'$ of the periodic wave, and the remaining eigenvalues have negative real parts, satisfying the spectral gap condition

$$\Re\left(\sigma_{L^2_{\mathrm{per}}(0,NT)}(\mathcal{A}[\phi]) \setminus \{0\}\right) \leq -\delta_N, \tag{3.1.6}$$

for some $\delta_N > 0$. As the eigenvalues of the Bloch operators $\mathcal{A}_\xi[\phi]$ depend continuously on $\xi$, it is not difficult to see that the spectral gap $\delta_N$ above tends to 0, as $N \to \infty$. For more information, see Section 1.3

For the periodic waves bifurcating from the dnoidal solutions of the NLS equation, the authors proved in [19] that some of these are spectrally stable to co-periodic perturbations. That this spectral stability corresponds to nonlinear stability was recently established in [67, Theorem 1]. The power of this result is that it reduces the problem of nonlinear stability for co-periodic perturbations to a spectral problem which, in turn, may be amenable to well-conditioned analytical and numerical methods. Furthermore, the proof of this result can be easily extended to subharmonic perturbations, leading to the following nonlinear stability result for diffusively spectrally stable periodic standing solutions of (3.1.1).

**Theorem 3.1.2.** *Let $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ be a $T$-periodic standing solution of* (3.1.1) *and fix $N \in \mathbb{N}$. Assume that $\phi$ is diffusively spectrally stable in the sense of Definition 1.3.1 and, for each $N \in \mathbb{N}$, take $\delta_N > 0$ such that* (3.1.6) *holds. Then for each $N \in \mathbb{N}$, $\phi$ is asymptotically stable to subharmonic $NT$-periodic perturbations. More precisely, for every $\delta \in (0, \delta_N)$ there exists an $\varepsilon = \varepsilon_\delta > 0$ and a constant $C = C_\delta > 0$ such that whenever $u_0 \in H^1_{\mathrm{per}}(0, NT)$ and $\|u_0 - \phi\|_{H^1(0,NT)} < \varepsilon$, then the* *solution $u$ of* (3.1.1) *with initial data $u(0) = u_0$ exists globally in time and satisfies*

$$\|u(\cdot,t) - \phi(\cdot - \gamma_\infty)\|_{H^1(0,NT)} \leq Ce^{-\delta t}\|u_0 - \phi\|_{H^1(0,NT)},$$

*for all $t > 0$, where $\gamma_\infty = \gamma_\infty(N)$ is some real constant.*

The key to the proof of Theorem 3.1.2, which is presented in [67] for $N = 1$, is a careful estimate of the resolvent operator, which allows one to apply the Gearhart-Prüss theorem and obtain an exponential decay rate for the semigroup generated by the linear operator $\mathcal{A}[\phi]$. Specifically, one shows that for each $\delta \in (0, \delta_N)$ there exists a constant $C = C_\delta > 0$ such that

$$\left\|e^{\mathcal{A}[\phi]t}\left(1 - \mathcal{P}_{0,N}\right) f\right\|_{H^1(0,NT)} \leq Ce^{-\delta t}\|f\|_{H^1(0,NT)} \tag{3.1.7}$$

for all $f \in H^1_{\text{per}}(0, NT)$, where here $\mathcal{P}_{0,N}$ is the rank-one spectral projection onto the $NT$-periodic kernel of $\mathcal{A}[\phi]$: see Remark 3.2.6(ii) for more details. Equipped with this linear exponential decay result, the remainder of the proof of Theorem 3.1.2 follows from standard nonlinear iteration arguments: for details, see [67]. We point out that using Strichartz-type estimates, the nonlinear stability result in Theorem 3.1.2 can be extended to $L^2$-subharmonic perturbations [4].

An important observation concerning Theorem 3.1.2 is that it lacks uniformity in $N$ in two (related) aspects. Indeed, note that both the exponential rate of decay, specified by $\delta$, as well as the allowable size of initial perturbations, specified by $\varepsilon = \varepsilon_\delta$, are controlled completely in terms of the size of the spectral gap $\delta_N > 0$ of the linearized operator $\mathcal{A}[\phi]$ in (3.1.6). Since $\delta_N \to 0$ as $N \to \infty$, it follows that both $\delta$ and $\varepsilon$ chosen in Theorem 3.1.2 necessarily tend to zero as $N \to \infty$ and that, consequently, the nonlinear stability result Theorem 3.1.2 is empty in the limit $N = \infty$. Note that at the linear level, the lack of uniformity in the allowable size of the initial perturbations is due to the fact that $C = C_\delta \to \infty$ as $\delta \to 0$ in (3.1.7).

In light of the above observations, it is therefore natural to ask if there is a way to obtain a stability result to subharmonic, i.e. $NT$-periodic, perturbations which is *uniform in $N$*. In such a result, one should require that both the rate of decay and allowable size of the initial perturbations are uniform in $N$, thus depending only on the background wave $\phi$. At the linear level, this would correspond to proving the existence of a non-negative function $g : (0, \infty) \to (0, \infty)$ with $g(t) \to 0$ as $t \to \infty$ such that an inequality of the form

$$\left\| e^{\mathcal{A}[\phi]t} \left(1 - \mathcal{P}_{0,N}\right) \right\|_{\mathcal{L}(L^2_{\text{per}}(0,NT))} \le g(t)$$

holds for all $N \in \mathbb{N}$ and $t > 0$. Our main result, stated below, shows that establishing such a uniform linear estimate is possible for the LLE (3.1.1), with polynomial rates of decay instead of exponential. Further, it gives additional insight into the long-time dynamics of subharmonic perturbations.

**Theorem 3.1.3** (Uniform Subharmonic Linear Asymptotic Stability)**.** *Suppose $\phi \in H^1_{\text{loc}}(\mathbb{R})$ is a $T$-periodic standing wave solution of* (3.1.1) *that is diffusively spectrally stable, in the sense of Definition 1.3.1. For each $N \in \mathbb{N}$ let*

$$\mathcal{P}_{0,N} : L^2_{\text{per}}(0, NT) \to \text{span}\{\phi'\}$$

*be the spectral projection of $L^2_{\text{per}}(0, NT)$ onto the $NT$-periodic kernel of $\mathcal{A}[\phi]$. Then there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ and $f \in L^1_{\text{per}}(0, NT) \cap L^2_{\text{per}}(0, NT)$ we have*

$$\left\| e^{\mathcal{A}[\phi]t}(1 - \mathcal{P}_{0,N})f \right\|_{L^2_{\text{per}}(0,NT)} \le C(1+t)^{-1/4} \|f\|_{L^1_{\text{per}}(0,NT) \cap L^2_{\text{per}}(0,NT)} \tag{3.1.8}$$

*valid for all $t > 0$. Furthermore, there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $f \in L^1_{\text{per}}(0, NT) \cap L^2_{\text{per}}(0, NT)$ there exists a $NT$-periodic function $\gamma_N(\cdot, t) = \gamma_N(\cdot, t; f)$*

$$\left\| \gamma_N(\cdot, t) - \frac{\left\langle \phi', \mathcal{P}_{0,N} f \right\rangle_{L^2_{\text{per}}(0,T)}}{\|\phi'\|^2_{L^2_{\text{per}}(0,T)}} \right\|_{L^2_{\text{per}}(0,NT)} \le C(1+t)^{-1/4} \|f\|_{L^1_{\text{per}}(0,NT) \cap L^2_{\text{per}}(0,NT)} \tag{3.1.9}$$

*and*

$$\left\| e^{\mathcal{A}[\phi]t} f - \phi' \gamma_N(\cdot, t) \right\|_{L^2_{\text{per}}(0,NT)} \le C(1+t)^{-3/4} \|f\|_{L^1_{\text{per}}(0,NT) \cap L^2_{\text{per}}(0,NT)}, \tag{3.1.10}$$

*for all $t > 0$.*

**Remark 3.1.4.** *More than above, we show in Section 3.5 below that the polynomial rates in Theorem 3.1.3 in fact provide* sharp *uniform rates of decay for subharmonic perturbations. Moreover, observe that the uniform polynomial rates of decay require control of the initial data in $L^1$. As we will see in our analysis, this is due to the fact that linear diffusion equation does not exhibit decay from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, but does from $L^1(\mathbb{R})$ to $L^2(\mathbb{R})$: see Remark 3.3.1 below.*

To interpret the above result, suppose that $\psi(x,t)$ is a solution of (3.1.1) with initial data $\psi(x,0) = \phi(x) + \varepsilon f(x)$ with $|\varepsilon| \ll 1$ and $f \in L^1_{\text{per}}(0, NT) \cap L^2_{\text{per}}(0, NT)$. By (3.1.8) it follows that, at the linear level, for $\varepsilon \neq 0$ sufficiently small, the solution $\psi$ essentially behaves for large time like

$$\psi(x,t) \approx \phi(x) + \varepsilon \mathcal{P}_{0,N} f(x) = \phi(x) + \varepsilon \frac{\left\langle \phi', \mathcal{P}_{0,N} f \right\rangle_{L^2_{\text{per}}(0,T)}}{\|\phi'\|^2_{L^2_{\text{per}}(0,T)}} \phi' \approx \phi\left( x + \varepsilon \frac{\left\langle \phi', \mathcal{P}_{0,N} f \right\rangle_{L^2_{\text{per}}(0,T)}}{\|\phi'\|^2_{L^2_{\text{per}}(0,T)}} \right),$$

corresponding to standard asymptotic (orbital) stability of $\phi$ with asymptotic phase. This recovers (again, at the linear level) the asymptotic stability result in Theorem 3.1.2, but now with asymptotic rates of decay which are uniform in $N$. Further than this, Theorem 3.1.3 implies that there exists a function $\gamma_N(x,t)$ which is $NT$-periodic in $x$ such that

$$\psi(x,t) \approx \phi(x) + \varepsilon \gamma_N(x,t) \phi'(x) \approx \phi(x + \varepsilon \gamma_N(x,t)), \quad t \gg 1,$$

with, by (3.1.10), a faster rate of convergence (in time), giving a refined insight into the long-time local dynamics near $\phi$ (described by a space-time dependent translational modulation) beyond the more standard asymptotic stability as in Theorem 3.1.2. As a consistency check, note that (3.1.9) implies that as $t \to \infty$ the function $\gamma_N(\cdot, t)$ tends to the asymptotic phase predicted by (3.1.8). As we will see, the incorporation of such a space-time dependent modulation function is key to our analysis and is precisely what allows us to obtain such uniform decay rates.

The key observation is that the bounds on the evolution semigroups and the linear rates of decay obtained in Theorem 3.1.3 on subharmonic perturbations are uniform in $N$. It turns out that these uniform decay rates are precisely the linear decay rates one obtains by considering the semigroup $e^{\mathcal{A}[\phi]t}$ as acting on $L^2(\mathbb{R})$, i.e., they agree exactly with the linear rates of decay for localized perturbations. That the decay rate to localized perturbations should uniformly control all subharmonic perturbations may be formally motivated by observing that, up to appropriate translations, a sequence of $NT$-periodic functions may converge as $N \to \infty$ to a function in $L^2(\mathbb{R})$ locally in space.

Based on the above comments, it shouldn't be surprising that the general methodology used for the proof of Theorem 3.1.3 is modeled off of the associated linear analysis to localized perturbations. The localized analysis, in turn, is largely based off the work [31] and, for completeness and to motivate the approach towards the proof of Theorem 3.1.3, we review the localized analysis in Section 3.3 below. In particular, we obtain the following result.

**Theorem 3.1.5** (Localized Linear Asymptotic Stability). *Suppose $\phi \in H^1_{\text{loc}}(\mathbb{R})$ is a $T$-periodic standing wave solution of* (3.1.1) *that is diffusively spectrally stable, in the sense of Definition 1.3.1. Then there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have*

$$\left\| e^{\mathcal{A}[\phi]t} f \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|f\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}, \tag{3.1.11}$$

*for all $t > 0$. Furthermore, there exists a constant $C > 0$ such that for each $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ there exists a function $\gamma(\cdot, t) = \gamma(\cdot, t; f)$ such that*

$$\|\gamma(\cdot, t)\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|f\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}$$

*and*

$$\left\| e^{\mathcal{A}[\phi]t} f - \phi' \gamma(\cdot, t) \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-3/4} \|f\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}, \tag{3.1.12}$$

*for all $t > 0$.*

The proofs of both Theorem 3.1.3 and Theorem 3.1.5 rely on a delicate decomposition of the semigroup $e^{\mathcal{A}[\phi]t}$ acting on the appropriate underlying space ($L^2_{\text{per}}(0, NT)$ and $L^2(\mathbb{R})$, respectively).

While the decomposition is similar in the two cases, the subharmonic result in Theorem 3.1.3 requires an additional level of decomposition which is not needed in the localized case. As we will see in Section 3.4 below, our proof of Theorem 3.1.3 connects to both the exponential decay result in Theorem 3.1.2 as well as the localized result in Theorem 3.1.5. Indeed, we will see that if one fixes $N \in \mathbb{N}$ then the (linear) exponential decay of $NT$-periodic perturbations follows naturally from our methodology. Further, by formally taking $N \to \infty$ we see that the result in Theorem 3.1.3 recovers the localized result in Theorem 3.1.5. For instance, we will see that, formally at least, the $NT$-periodic modulation function $\gamma_N$ in Theorem 3.1.3 satisfies

$$\lim_{N \to \infty} \gamma_N(x,t) = \gamma(x,t),$$

where here $\gamma$ is the localized modulation function from Theorem 3.1.5. In this way, our work both expands and unifies several previous works in the literature.

**Remark 3.1.6.** *During our proof of Theorem 3.1.3, we will also see how the techniques presented provide exponential decay results of the form* (3.1.7) *with a constant* $C > 0$ *which is* uniform *in* $N$*. This extends the key linear estimate* (3.1.7) *used in [67] to establish Theorem 3.1.2.*

We also emphasize that our methodology used in the proofs of Theorem 3.1.3 and Theorem 3.1.5 is quite general and applies more broadly than for the LLE (3.1.1). As mentioned previously, our arguments are motivated by the recent work [31] which, in turn, was based on a sequence of previous works [34, 39, 37, 36, 29, 30, 2] by the same authors, all of which were eventually based on the seminal work of Schneider [59, 60]. In fact, our work relies on only a few key features of the linearized operator $\mathcal{A}[\phi]$. Namely, Theorems 3.1.3 and Theorem 3.1.5 continue to hold provided the following properties are satisfied:

(i) The wave $\phi$ is diffusively spectrally stable, as defined in Definition 1.3.1.

(ii) The operator $\mathcal{A}[\phi]$ generates $C^0$-semigroups on $L^2(\mathbb{R})$ and $L^2_{\text{per}}(0, NT)$, and for each $\xi \in [-\pi/T, \pi/T)$ the Bloch operators $\mathcal{A}_\xi[\phi]$ generate $C^0$-semigroups on $L^2_{\text{per}}(0, T)$.

(iii) There exist positive constants $\mu_0$ and $C_0$ such that for each $\xi \in [-\pi/T, \pi/T)$ the Bloch resolvent operators satisfy

$$\|(i\mu - \mathcal{A}_\xi[\phi])^{-1}\|_{\mathcal{L}(L^2_{\text{per}}(0,T))} \leq C_0, \quad \text{for all } |\mu| > \mu_0. \tag{3.1.13}$$

Consequently, our work sets forth a general methodology for establishing analogous (linear) results to Theorem 3.1.3 and Theorem 3.1.5 in more general contexts. Further, note that the conditions (i)-(iii) above are slightly more general than the corresponding assumptions used in [31]. For our analysis of the LLE (3.1.1), the first property above is the main assumption, that we know it holds at least for the periodic waves $\phi_\mu$ given by (3.1.5), and the other two properties are proved in Section 3.2.

Of course, it is natural to ask if our linear results can be extended to a result pertaining to the nonlinear dynamics of the LLE (3.1.1). Although the linear estimate (3.1.11) suggests algebraic decay of $L^2(\mathbb{R})$ perturbations, we will see in Remark 5.4.3 below that the $(1+t)^{-1/4}$ decay is insufficient to close the corresponding nonlinear iteration scheme. The refined linear estimate (3.1.12), however, suggests that the incorporation of a spatio-temporal phase modulation may allow for sufficiently fast decay. Unfortunately, as we will see in Section 3.6 and the following chapters, the incorporation of such a spatially-dependent modulation function introduces a loss of derivatives in the nonlinear iteration scheme. Such a phenomena is well known in the context of reaction diffusion equations and systems of conservation laws, where the loss of derivatives can be compensated by a nonlinear damping effect which slaves high Sobolev norms to low Sobolev norms – see, for example, Remark 3.6.1 and Chapter 4 below. However, such nonlinear damping techniques rely heavily on the damping in the governing evolution equation to correspond to the highest-order spatial derivative present. In the case of the LLE, unfortunately, damping appears as the lowest-order derivative and hence one has no hope of regaining derivatives through such nonlinear damping estimates. Circumventing this issue, in the case of localized perturbations, will be taken up in Chapter 5.

The outline for this chapter is as follows. In Section 3.2, we collect the properties of the Bloch operators associated to $\mathcal{A}[\phi]$ required for our analysis. We describe their spectral properties, then establish the existence and basic decay properties of the corresponding Bloch semigroups. Section 3.3 is dedicated to the proof of the localized result Theorem 3.1.5, which in turn serves as motivation for our proof of Theorem 3.1.3, which is presented in Section 3.4. In Section 3.5, we present a technical bound which establishes that the decay rates for localized perturbations in Theorem 3.1.5 in fact provide *sharp* uniform decay rates for subharmonic perturbations. The proof of this key bound is given in Appendix 3.A. Finally, in Section 3.6 we describe the mathematical challenges encountered in establishing the corresponding nonlinear stability of diffusively spectrally stable periodic standing solutions of the LLE (3.1.1). Throughout our work, we aim to make clear the ways in which our analysis unifies and expands previous works, as well as its ability to be generalized to other contexts.

## 3.2 Spectral Preparation & Properties of Bloch Semigroups

In this section, we prove several preliminary results. In particular, we establish some spectral properties and semigroup estimates for the Bloch operators obtained from the LLE (3.1.1).

Assuming that $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a diffusively spectrally stable $T$-periodic stationary solution of the profile equation (3.1.2), we now consider the linear operators $\mathcal{A}[\phi]$ and $\mathcal{A}_\xi[\phi]$ found from the LLE (3.1.1). Recall that $\mathcal{A}[\phi]$ is given by (3.1.4) from which we find the Bloch operators

$$\mathcal{A}_\xi[\phi] = -I + \mathcal{J}\mathcal{L}_\xi[\phi], \tag{3.2.1}$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_\xi[\phi] = \begin{pmatrix} -\beta(\partial_x + i\xi)^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta(\partial_x + i\xi)^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}.$$

Our first lemma summarizes the spectral properties of the Bloch operators $\mathcal{A}_\xi[\phi]$ which directly follow from the definition of diffusive spectral stability.

**Lemma 3.2.1** (Spectral Preparation)**.** *Suppose $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a diffusively spectrally stable $T$-periodic stationary solution of the LLE* (3.1.1)*. Then the following properties hold.*

*(i) For any fixed $\xi_0 \in (0, \pi/T)$, there exists a positive constant $\delta_0$ such that*

$$\Re\sigma(\mathcal{A}_\xi[\phi]) < -\delta_0,$$

*for all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.*

*(ii) There exist positive constants $\xi_1$, $\delta_1$, and $d$ such that for any $|\xi| < \xi_1$ the spectrum of $\mathcal{A}_\xi[\phi]$ decomposes into two disjoint subsets*

$$\sigma(\mathcal{A}_\xi[\phi]) = \sigma_-(\mathcal{A}_\xi[\phi]) \cup \sigma_0(\mathcal{A}_\xi[\phi]),$$

*with the following properties:*

*(a) $\Re\sigma_-(\mathcal{A}_\xi[\phi]) < -\delta_1$ and $\Re\sigma_0(\mathcal{A}_\xi[\phi]) > -\delta_1$;*

*(b) the set $\sigma_0(\mathcal{A}_\xi[\phi])$ consists of a single negative eigenvalue $\lambda_c(\xi)$ which is analytic in $\xi$ and expands as*

$$\lambda_c(\xi) = ia\xi - d\xi^2 + O(\xi^3),$$

*for $|\xi| \ll 1$ for some $a \in \mathbb{R}$ and $d > 0$;*

*(c) the eigenfunction associated to $\lambda_c(\xi)$ is analytic near $\xi = 0$ and expands as*

$$\Phi_\xi(x) = \phi'(x) + O(\xi),$$

*where $\phi'$ is the derivative of the $T$-periodic solution $\phi$.*

*Proof.* The first part is an immediate consequence of the properties (i) and (ii) in the Definition 1.3.1. To prove the second part, observe that since $\lambda = 0$ is a simple, isolated eigenvalue of $\mathcal{A}_0[\phi]$, and since $\mathcal{A}_\xi[\phi]$ depend continuously on $\xi$, standard spectral perturbation theory implies the continuous dependence on $\xi$ of the eigenvalue $\lambda_c(\xi)$ and of its associated eigenvector $\Phi_\xi(x)$. $\square$

**Remark 3.2.2.** *In the case where the background wave $\phi$ is even (up to translation), a simple symmetry argument implies that $a = 0$ in the expansion of $\lambda_c(\xi)$ above. Note that the diffusively spectrally stable solutions* (3.1.5) *constructed and studied via bifurcation theory in [11, 12] are all even. For more general waves, however, one may have $a \neq 0$.*

Our next result establishes that the linearized operator $\mathcal{A}[\phi]$ and its associated Bloch operators generate $C^0$-semigroups. The proof is elementary and relies on a decomposition of these operators into a constant coefficient operator plus a bounded perturbation. Specifically, we establish the following result (see also [67, Lemma 1]).

**Lemma 3.2.3.** *Assume that $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a $T$-periodic solution of the stationary LLE* (3.1.2). *The linear operator $\mathcal{A}[\phi]$ acting in either $L^2(\mathbb{R})$ or $L^2_{\mathrm{per}}(0, NT)$ generates a $C^0$ semigroup. Similarly, for each $\xi \in [-\pi/T, \pi/T)$ the Bloch operators $\mathcal{A}_\xi[\phi]$ acting in $L^2_{\mathrm{per}}(0,T)$ generate $C^0$-semigroups.*

*Proof.* The proofs being the same in the three cases, we only consider the Bloch operators $\mathcal{A}_\xi[\phi]$ acting in $L^2_{\mathrm{per}}(0,T)$. From (3.2.1) we see that $\mathcal{A}_\xi[\phi]$ is a bounded perturbation of the operator

$$\mathcal{A}^0_\xi = -\beta(\partial_x + i\xi)^2 \mathcal{J}.$$

Since bounded perturbations of generators of $C^0$-semigroups also generate a $C^0$-semigroups [54, Theorem 1.1, Section 3.1], it remains to prove the result for the operator $\mathcal{A}^0_\xi$. This operator is closed on $L^2_{\mathrm{per}}(0,T)$ with domain $H^2_{\mathrm{per}}(0,T)$ and has constant coefficients. Using Fourier analysis, it is then straightforward to check that its spectrum lies on the imaginary axis $i\mathbb{R}$ and that for any complex number $\lambda$ in its resolvent set the norm of the resolvent operator is given by

$$\|(\lambda - \mathcal{A}^0_\xi)^{-1}\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} = \frac{1}{\mathrm{dist}(\lambda, \sigma(\mathcal{A}^0_\xi))}.$$

Consequently, for any complex number $\lambda$ with $\Re\lambda > 0$ we have

$$\|(\lambda - \mathcal{A}_\xi^0)^{-1}\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} = \frac{1}{\Re\lambda}.$$

Together with the Hille-Yosida theorem (e.g., see [42, Chapter IX.2]) this implies that $\mathcal{A}_\xi^0$ generates a $C^0$-semigroup which proves the lemma. □

The next step consists in proving the resolvent estimate (3.1.13) for the Bloch operators $\mathcal{A}_\xi[\phi]$ given by the LLE (3.1.1)[4].

**Lemma 3.2.4.** *Suppose $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a diffusively spectrally stable $T$-periodic stationary solution of the LLE* (3.1.1). *There exist positive constants $\mu_0$ and $C_0$ such that for each $\xi \in [-\pi/T, \pi/T)$ the Bloch resolvent operators satisfy*

$$\|(i\mu - \mathcal{A}_\xi[\phi])^{-1}\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_0, \quad \text{for all } |\mu| > \mu_0. \tag{3.2.2}$$

*Proof.* For $\xi = 0$ this result has been proved in [67, Proposition 1]. It can be easily extended to $\xi \in [-\pi/T, \pi/T)$ by replacing the (spatial) Fourier frequency $k$ in their expression of the linear operator by $k + \xi$. □

Combining the result in this lemma with the spectral properties in Lemma 3.2.1, we obtain the following estimates for the Bloch semigroups $e^{\mathcal{A}_\xi[\phi]t}$.

**Proposition 3.2.5.** *Suppose $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a diffusively spectrally stable $T$-periodic stationary solution of the LLE* (3.1.1). *Then the following properties hold.*

*(i) For any fixed $\xi_0 \in (0, \pi/T)$, there exist positive constants $C_0$ and $\eta_0$ such that*

$$\left\|e^{\mathcal{A}_\xi[\phi]t}\right\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_0 e^{-\eta_0 t},$$

*for all $t \geq 0$ and all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.*

[4]For the periodic waves $\phi_\mu$ given by (3.1.5) this result can be easily proved using perturbation arguments, since the operators are in this case small bounded perturbations of operators with constant coefficients.

*(ii) With $\xi_1$ chosen as in Lemma 3.2.1 (ii), there exist positive constants $C_1$ and $\eta_1$ such that for any $|\xi| < \xi_1$, if $\Pi(\xi)$ is the spectral projection onto the (one-dimensional) eigenspace associated to the eigenvalue $\lambda_c(\xi)$ given by Lemma 3.2.1 (ii), then*

$$\left\| e^{\mathcal{A}_\xi[\phi]t} \left(I - \Pi(\xi)\right) \right\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \leq C_1 e^{-\eta_1 t},$$

*for all $t \geq 0$.*

*Proof.* We use the Gearhart-Prüss theorem[5] to prove the result. The absence of purely imaginary spectrum for the Bloch operators $\mathcal{A}_\xi[\phi]$ for $\xi \neq 0$, implies that the resolvent estimate (3.2.2) holds for any purely imaginary number $i\mu$, uniformly for all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$, for some fixed $\xi_0 \in (0, \pi/T)$. The Gerhart-Prüss theorem then implies the result in the first part of the lemma. The second part follows in the same way, noting that the operator $\mathcal{A}_\xi[\phi](I - \Pi(\xi))$ has no purely imaginary spectrum for any $\xi \in [-\pi/T, \pi/T)$. □

**Remark 3.2.6.** *(i) In the context of the LLE, the resolvent estimate* (3.2.2) *actually holds for any complex number $\lambda = \delta + i\mu$ with $\delta > -1$ and $|\mu| > \mu_\delta$, for some $\mu_\delta > 0$, which is the analogue for $\xi \neq 0$ of the result obtained in [67, Proposition 1] for $\xi = 0$. As a consequence, one can better characterize the exponential rates of decay in Proposition 2.6 by showing that*

$$0 < \eta_0 < -\max\left\{\Re\sigma(\mathcal{A}_\xi[\phi]);\ \xi \in [-\pi/T, \pi/T),\ |\xi| > \xi_0\right\},$$

*and*

$$0 < \eta_1 < -\max\left\{\Re\left(\sigma(\mathcal{A}_\xi[\phi]) \setminus \{\lambda_c(\xi)\}\right);\ |\xi| < \xi_1\right\}.$$

*For our purposes, however, we do not need this more precise characterization.*

*(ii) Together with the Floquet-Bloch theory for subharmonic perturbations from Section 1.3.4, the result in Proposition 3.2.5 gives the following estimate for the $C^0$-semigroup generated by*

[5] Technically, we are using a slight extension of the Gearhart-Prüss theorem, requiring uniform boundedness of the resolvent on the imaginary axis as opposed to a half plane. See [16, Corollary 2] for a proof of this extension.

*$\mathcal{A}[\phi]$ when acting in $L^2_{\text{per}}(0, NT)$,*

$$\left\| e^{\mathcal{A}[\phi]t}(1 - \mathcal{P}_{0,N})f \right\|_{L^2_{\text{per}}(0,NT)} \leq C_N e^{-d_N t},$$

*for all $t \geq 0$, where $\mathcal{P}_{0,N}$ is the spectral projection of $L^2_{\text{per}}(0, NT)$ onto the $NT$-periodic kernel of $\mathcal{A}[\phi]$ in Theorem 3.1.3. The above remark implies that $d_N \in (0, \delta_N)$, where $\delta_N$ is given by* (3.1.6). *This is the exponential decay rate for the semigroup generated by $\mathcal{A}[\phi]$ mentioned in the introduction to this chapter and required to prove the result in Theorem 3.1.2.*

## 3.3 Linear Asymptotic Modulational Stability to Localized Perturbations

Recall from the introduction to this chapter that our goal is to obtain uniform rates of decay for subharmonic perturbations of a given $T$-periodic stationary solution of (3.1.1). Our argument, which will be presented in Section 3.4 below, is largely motivated by and modeled after the stability analysis to perturbations that are localized on $\mathbb{R}$. As such, we first consider the case of localized perturbations and present the proof of Theorem 3.1.5. We emphasize that the proof of Theorem 3.1.5 follows the general methodology introduced in [31] for the stability of periodic waves in general conservation or balance laws to classes of both localized and non-localized[6] perturbations. Nevertheless, we briefly outline the argument, for motivational purposes, as it applies to (3.1.1).

Following [31], the general strategy of the proof of Theorem 3.1.5 is to use the Bloch transform to obtain estimates on the semigroup $e^{\mathcal{A}[\phi]t}$ from estimates on the Bloch semigroups $e^{\mathcal{A}_\xi[\phi]t}$ in Proposition 3.2.5. To this end, our goal is to decompose the semigroup $e^{\mathcal{A}[\phi]t}$ as

$$e^{\mathcal{A}[\phi]t} = \text{“Critical Part”} + \text{“Exponentially Damped Part”},$$

where, owing to Lemma 3.2.1, the “critical part” should be dominated by the translation mode $\phi'$. Note throughout this section, we adopt the notation $A \lesssim B$ to mean there exists a constant $C > 0$

[6]In [31] the authors results cover the case when the perturbation slightly changes the phase at infinity of the underlying wave. We do not consider such an extension here.

such that $A \leq CB$.

Let $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We begin by decomposing $e^{\mathcal{A}[\phi]t}$ into low-frequency and high-frequency components: with $\xi_1$ defined as in Proposition 3.2.5, let $\rho$ be a smooth cutoff function with $\rho(\xi) = 1$ for $|\xi| < \xi_1/2$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$ and use (1.3.9) to write

$$\begin{aligned} e^{\mathcal{A}[\phi]t}v(x) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\check{v}(\xi,x)d\xi + \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}(1-\rho(\xi))e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\check{v}(\xi,x)d\xi \\ &=: S_{lf}(t)v(x) + S_{hf}(t)v(x), \end{aligned} \tag{3.3.1}$$

where $S_{lf}$ and $S_{hf}$ denote the low- and high-frequency components of the solution operator $e^{\mathcal{A}[\phi]t}$, respectively. According to Lemma 3.2.1, the spectrum of the Bloch operators $\mathcal{A}_\xi[\phi]$ have a uniform spectral gap on the support of $(1-\rho(\xi))$ and hence, by Parseval's identity (1.3.8) and Proposition 3.2.5, there exists a constant $\eta > 0$ such that

$$\left\|S_{hf}(t)v\right\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left\|(1-\rho(\xi))e^{\mathcal{A}_\xi[\phi]t}\check{v}(\xi,x)\right\|_{L^2(0,T)}^2 d\xi \lesssim e^{-\eta t}\|v\|_{L^2(\mathbb{R})} \tag{3.3.2}$$

valid for all $v \in L^2(\mathbb{R})$. It thus remains to study the low-frequency component of the solution operator.

To this end, we further decompose $S_{lf}(t)$ into the contribution from the critical mode near $(\lambda,\xi) = (0,0)$ and the contribution from the low-frequency spectrum bounded away from $\lambda = 0$. Accordingly, for each $|\xi| < \xi_1$ let $\Pi(\xi)$ be the spectral projection onto the critical mode of $\mathcal{A}_\xi[\phi]$ as defined in Proposition 3.2.5, and note it is given explicitly via

$$\Pi(\xi)g(x) = \left\langle\widetilde{\Phi}_\xi, g\right\rangle_{L^2(0,T)}\Phi_\xi(x), \tag{3.3.3}$$

where $\widetilde{\Phi}_\xi$ is the element in the kernel of the adjoint $\mathcal{A}_\xi[\phi]^* - \overline{\lambda_c(\xi)}I$ satisfying $\left\langle\widetilde{\Phi}_\xi, \Phi_\xi\right\rangle_{L^2(0,T)} = 1$. We can thus decompose $S_{lf}(t)$ as

$$S_{lf}(t)v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\Pi(\xi)\check{v}(\xi,x)d\xi$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}(1-\Pi(\xi))\check{v}(\xi,x)d\xi \\
=:\ & S_c(t)v(x)+\widetilde{S}_{lf}(t)v(x).
\end{aligned}
\tag{3.3.4}
$$

Using Parseval's identity (1.3.8) and Proposition 3.2.5 again, it follows that, by possibly choosing $\eta > 0$ above smaller,

$$
\left\|\widetilde{S}_{lf}(t)v\right\|_{L^2(\mathbb{R})} \lesssim e^{-\eta t}\|v\|_{L^2(\mathbb{R})}
\tag{3.3.5}
$$

valid for all $v \in L^2(\mathbb{R})$.

For the critical component $S_c(t)$ of the solution operator, note that for any $v \in L^2(\mathbb{R})$ we have

$$
\begin{aligned}
e^{\mathcal{A}_\xi[\phi]t}\Pi(\xi)\check{v}(\xi,x) &= e^{\lambda_c(\xi)t}\Phi_\xi(x)\left\langle\widetilde{\Phi}_\xi,\check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} \\
&= e^{\lambda_c(\xi)t}\left(\phi'+i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\right)\left\langle\widetilde{\Phi}_\xi,\check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)},
\end{aligned}
$$

where we note by Lemma 3.2.1 that

$$
\sup_{x\in\mathbb{R}}\left(\sup_{\xi\in[-\pi/T,\pi/T)}\left|\rho(\xi)\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\right|\right) \lesssim 1.
\tag{3.3.6}
$$

We may thus decompose $S_c$ further as

$$
\begin{aligned}
S_c(t)v(x) &= \phi'(x)\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)e^{i\xi x+\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi,\check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}d\xi \\
&\quad+\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)e^{i\xi x+\lambda_c(\xi)t}i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\left\langle\widetilde{\Phi}_\xi,\check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}d\xi \\
&= \phi'(x)s_p(t)v(x)+\widetilde{S}_c(t)v(x).
\end{aligned}
\tag{3.3.7}
$$

Now, observe that by definition (see (1.3.7)) we have

$$
\begin{aligned}
\left\langle\widetilde{\Phi}_\xi,\check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} &= \int_0^T\overline{\widetilde{\Phi}_\xi(x)}\sum_{\ell\in\mathbb{Z}}e^{2\pi i\ell x/T}\hat{v}(\xi+2\pi\ell/T)\,dx \\
&= \sum_{\ell\in\mathbb{Z}}\hat{v}(\xi+2\pi\ell/T)\int_0^T\overline{\widetilde{\Phi}_\xi(x)}e^{2\pi i\ell x/T}dx
\end{aligned}
$$

$$= \sum_{\ell\in\mathbb{Z}} \hat{v}(\xi + 2\pi\ell/T)\overline{\hat{\widetilde{\Phi}}_\xi(2\pi\ell/T)}$$

and hence, since $|\hat{v}(z)| \le \|v\|_{L^1(\mathbb{R})}$ for all $z \in \mathbb{R}$, we have

$$\begin{aligned}
\rho(\xi)\left|\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right|^2 &\le \rho(\xi)\left(\sum_{\ell\in\mathbb{Z}} |\hat{v}(\xi+2\pi\ell/T)|\left|\hat{\widetilde{\Phi}}_\xi(2\pi\ell/T)\right|\right)^2 \\
&\le \rho(\xi)\|v\|^2_{L^1(\mathbb{R})}\left(\sum_{\ell\in\mathbb{Z}}(1+|\ell|^2)^{1/2}\left|\hat{\widetilde{\Phi}}_\xi(2\pi\ell/T)\right|(1+|\ell|^2)^{-1/2}\right)^2 \\
&\lesssim \|v\|^2_{L^1(\mathbb{R})} \sup_{\xi\in[-\pi/T,\pi/T)}\left(\rho(\xi)\left\|\widetilde{\Phi}_\xi\right\|^2_{H^1_{\mathrm{per}}(0,T)}\right),
\end{aligned} \tag{3.3.8}$$

where the final inequality follows by the Cauchy-Schwarz inequality. Using Parseval's identity (1.3.8) it follows that the phase shift component of $S_c$ satisfies

$$\begin{aligned}
\left\|s_p(t)v\right\|_{L^2(\mathbb{R})} &= \left(\frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\left\|\rho(\xi)e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right\|^2_{L^2(0,T)} d\xi\right)^{1/2} \\
&\lesssim \|e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})} \\
&\lesssim (1+t)^{-1/4}\|v\|_{L^1(\mathbb{R})}
\end{aligned} \tag{3.3.9}$$

and, similarly,

$$\left\|\widetilde{S}_c(t)v\right\|_{L^2(\mathbb{R})} \lesssim \|\xi e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-3/4}\|v\|_{L^1(\mathbb{R})}, \tag{3.3.10}$$

where the bounds on $\|e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}$ and $\|\xi e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}$ follow from an elementary scaling analysis.

In summary, for each $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have the decomposition

$$e^{\mathcal{A}[\phi]t}v(x) = \phi'(x)s_p(t)v(x) + \widetilde{S}_c(t)v(x) + \widetilde{S}_{lf}(t)v(x) + S_{hf}(t)v(x), \tag{3.3.11}$$

where the operators $s_p(t)$ and $\widetilde{S}_c(t)$ are defined in (3.3.7), and where the operators $\widetilde{S}_{lf}(t)$ and $S_{hf}(t)$ are defined in (3.3.4) and (3.3.1), respectively. Recalling the estimates (3.3.9)-(3.3.10), valid for all $t \geq 0$, as well as the exponential decay estimates (3.3.5) and (3.3.2) on $\widetilde{S}_{lf}(t)$ and $S_{hf}(t)$, respectively, the proof of Theorem 3.1.5 follows by setting $\gamma(x,t) := \left(s_p(t)v\right)(x)$.

**Remark 3.3.1.** *Note that to obtain $L^2 \to L^2$ bounds on $s_p$ and $\widetilde{S}_c$ above, one would use in place of* (3.3.8) *the slightly sharper estimate*

$$\rho(\xi)\left|\left\langle\widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right|^2 \leq \rho(\xi)\left\|\widetilde{\Phi}_\xi\right\|^2_{L^2(0,T)} \|\check{v}(\xi,\cdot)\|^2_{L^2(0,T)}$$

*which, by Parseval, would then lead to the bounds*

$$\|s_p(t)v\|_{L^2(\mathbb{R})} \lesssim \|e^{-d\xi^2 t}\|_{L^\infty_\xi(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} \lesssim \|v\|_{L^2(\mathbb{R})}$$

*and*

$$\|\widetilde{S}_c(t)v\|_{L^2(\mathbb{R})} \lesssim \|\xi e^{-d\xi^2 t}\|_{L^\infty_\xi(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-1/2}\|v\|_{L^2(\mathbb{R})}$$

*In particular, since $s_p(t)$ does not exhibit decay from $L^2$ to $L^2$, the final decomposition* (3.3.11) *implies only a bounded linear stability from $L^2$ to $L^2$. The faster polynomial rates of decay in* (3.3.9)-(3.3.10) *rely on being able to control the initial perturbation in $L^1$ as well, and introduces diffusive rates of decay of perturbations.*

Finally, we end our study of the localized analysis by describing at a finer level the long-time dynamics of the modulation function $\gamma$ in Theorem 3.1.5. Note that from the explicit form of the phase operator $s_p(t)$ defined in (3.3.7) it is natural to expect that for a given $v \in L^2(\mathbb{R})$ the function $s_p(t)v$ should be well approximated (for at least large time) by the function

$$(w(t)v)(x) := \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x+(ia\xi-d\xi^2)t}\left\langle\widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} d\xi. \tag{3.3.12}$$

Precisely, following the techniques from above we have the bound

$$\left\|s_p(t)v - w(t)v\right\|_{L^2(\mathbb{R})} = \left\|\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}\rho(\xi)\left(e^{\lambda_c(\xi)t} - e^{(ia\xi-d\xi^2)t}\right)\left\langle\widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} d\xi\right\|_{L^2(\mathbb{R})}$$

$$\begin{aligned}
&\lesssim \left\| e^{-d\xi^2 t}\left( e^{(\lambda_c(\xi)-(ia\xi-d\xi^2))t}-1\right)\right\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})}\\
&\lesssim \left\| e^{-d\xi^2 t}\xi^3 t\right\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})}\\
&\lesssim (1+t)^{-3/4}\|v\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Noting that the above $w(x,t) := (w(t)v)(x)$ defined in (3.3.12) is the unique solution of the linear diffusion IVP

$$\begin{cases} w_t = aw_x + dw_{xx} \\ w(x,0) = s_p(0)v(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} d\xi. \end{cases} \tag{3.3.13}$$

posed on $L^2(\mathbb{R})$, this establishes the following result concerning the behavior of the modulation function $\gamma(x,t)$ from Theorem 3.1.5.

**Corollary 3.3.2** (Asymptotic Modulational Behavior)**.** *Under the hypotheses of Theorem 3.1.5, for each $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the modulation function $\gamma$ satisfies the estimate*

$$\|\gamma(\cdot,t) - w(\cdot,t)\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-3/4}\|v\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}$$

*valid or all $t > 0$, where here $w$ is the solution to the linear heat equation*

$$w_t = -i\lambda_c'(0)w_x - \frac{1}{2}\lambda_c''(0)w_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

*with initial data prescribed as in* (3.3.13).

**Remark 3.3.3.** *To place the final calculations above in a broader context, we note that the PDE in* (3.3.13) *corresponds to the Whitham modulation equation associated to* (3.1.1)*. Following the work in [31, Appendix B] it is possible to show through a formal multiple scales analysis that an* *approximate solution to the profile equation* (3.1.2) *is given by*

$$\psi(x,t) \approx \widetilde{\psi}\left(\Psi(x,t)\right)$$

*where the wave number* $k := \Psi_x$ *satisfies the Whitham equation*

$$k_t = -i\lambda_c'(0)k_x - \frac{1}{2}\lambda_c''(0)k_{xx}.$$

*This suggests that the large time modulational behavior should be governed by a solutions of the heat equation, which is precisely what is made rigorous through Corollary 3.3.2. For more information on dynamical predictions of Whitham modulation equations, see [13, 31].*

## 3.4 Linear Asymptotic Modulational Stability to Subharmonic Perturbations

Motivated by the analysis in Section 3.3, we now strive to obtain decay rates on the semigroup $e^{\mathcal{A}[\phi]t}$ acting on classes of subharmonic perturbations which are uniform in $N$. As we will see, the analysis is based on a decomposition of the solution operator $e^{\mathcal{A}[\phi]t}$ which is largely motivated by the analysis in Section 3.3. One key difference, however, is that in the subharmonic case, the eigenvalue $\lambda = 0$ is isolated from the remaining $NT$-periodic eigenvalues of $\mathcal{A}[\phi]$, which leads to a slightly different decomposition of the solution operator near $(\lambda, \xi) = (0,0)$ than used in Section 3.3 above.

For each $N \in \mathbb{N}$ and each $p \geq 1$, we set, for notational convenience,

$$L^p_N := L^p_{\text{per}}(0, NT).$$

Further, throughout the remainder of this chapter, we will use the previously introduced notation $A \lesssim B$ to indicate there exists a constant $C > 0$ which is *independent of* $N$ such that $A \leq CB$. Now, for fixed $N \in \mathbb{N}$, let $f \in L^2_N$ and recall from (1.3.16) that the action of the semigroup $e^{\mathcal{A}[\phi]t}$ on $f$ can be represented through the use of the Bloch transform as[7]

$$e^{\mathcal{A}[\phi]t} f(x) = \frac{1}{NT} \sum_{\xi \in \Omega_N} e^{i\xi x} e^{\mathcal{A}_\xi[\phi]t} \mathcal{B}_T(f)(\xi, x),$$

where $\Omega_N \subset [-\pi/T, \pi/T)$ is defined in (1.3.10) as the set of Bloch frequencies corresponding to $NT$-periodic perturbations and $\mathcal{B}_T(f)$, defined in (1.3.13), denotes the $T$-periodic Bloch transform of the $NT$-periodic function $f$.

Following the general procedure in Section 3.3, we begin by decomposing the subharmonic solution operator into low-frequency and high-frequency parts. Letting $\rho$ be a smooth cut-off function as in Section 3.3, note for all $f \in L^2_N$ we have

$$\begin{aligned} e^{\mathcal{A}[\phi]t} f(x) &= \frac{1}{NT} \sum_{\xi \in \Omega_N} \rho(\xi) e^{i\xi x} e^{\mathcal{A}_\xi[\phi]t} \mathcal{B}_T(f)(\xi, x) + \frac{1}{NT} \sum_{\xi \in \Omega_N} (1 - \rho(\xi))\, e^{i\xi x} e^{\mathcal{A}_\xi[\phi]t} \mathcal{B}_T(f)(\xi, x) \\ &=: S_{lf,N}(t) f(x) + S_{hf,N}(t) f(x). \end{aligned} \tag{3.4.1}$$

To estimate the high-frequency component, we use the discrete Parseval identity (1.3.15) to get

$$\begin{aligned} \|S_{hf,N}(t) f\|^2_{L^2_N} &= \frac{1}{NT^2} \sum_{\xi \in \Omega_N} \|(1 - \rho(\xi)) e^{\mathcal{A}_\xi[\phi]t} \mathcal{B}_T(f)(\xi, \cdot)\|^2_{L^2(0,T)} \\ &\leq \frac{1}{NT^2} \sum_{\xi \in \Omega_N} (1 - \rho(\xi))^2 \|e^{\mathcal{A}_\xi[\phi]t}\|^2_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \|\mathcal{B}_T(f)(\xi, \cdot)\|^2_{L^2(0,T)}. \end{aligned}$$

From Proposition 3.2.5, it follows from Lemma 3.2.1 and Proposition 3.2.5 that there exists a constant $\eta > 0$ such that

$$\max_{\xi \in \Omega_N} (1 - \rho(\xi))^2 \|e^{\mathcal{A}_\xi[\phi]t}\|_{\mathcal{L}(L^2_{\mathrm{per}}(0,T))} \lesssim e^{-\eta t}$$

[7]This representation formula is the analogue of (1.3.9) used in Section 3.3.

yielding the exponential decay estimate

$$\|S_{hf,N}(t)f\|_{L^2_N} \lesssim e^{-\eta t}\left(\frac{1}{NT^2}\sum_{\xi\in\Omega_N}\|\mathcal{B}_T(f)(\xi,\cdot)\|^2_{L^2(0,T)}\right)^{1/2} = e^{-\eta t}\|f\|_{L^2_N}, \tag{3.4.2}$$

where the last equality again follows by Parseval's identity (1.3.15).

Continuing on, to study the low-frequency component, let $\Pi(\xi)$ be the rank-one spectral projection onto the critical mode of $\mathcal{A}_\xi[\phi]$ from Proposition 3.2.5 and note that $S_{lf,N}(t)$ can be further decomposed into the contribution from the critical mode near $(\lambda,\xi)=(0,0)$ and the contribution from the low-frequency spectrum bounded away from $\lambda=0$ via

$$\begin{aligned} S_{lf,N}(t)f(x) &= \frac{1}{NT}\sum_{\xi\in\Omega_N}\rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}\Pi(\xi)\mathcal{B}_T(f)(\xi,x) \\ &\qquad + \frac{1}{NT}\sum_{\xi\in\Omega_N}\rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi[\phi]t}(1-\Pi(\xi))\mathcal{B}_T(f)(\xi,x) \\ &=: S_{c,N}(t)f(x)+\widetilde{S}_{lf,N}(t)f(x). \end{aligned} \tag{3.4.3}$$

Using Parseval's identity (1.3.15) and Proposition 3.2.5 we know by possibly choosing $\eta>0$ above even smaller, that

$$\|\widetilde{S}_{lf,N}(t)f\|_{L^2_N} \lesssim e^{-\eta t}\|f\|_{L^2_N}. \tag{3.4.4}$$

For the critical component of $S_{c,N}(t)$, note that by Lemma 1.3.5 the $\xi=0$ term[8] can be identified as

$$\frac{1}{NT}\Pi(0)\mathcal{B}_T(f)(0,x) = \frac{1}{NT}\phi'(x)\left\langle\widetilde{\Phi}_0,\mathcal{B}_T(f)(0,\cdot)\right\rangle_{L^2(0,T)} = \frac{1}{N}\phi'(x)\left\langle\widetilde{\Phi}_0,f\right\rangle_{L^2_N}.$$

Since (3.3.3) and Lemma 3.2.1 imply the projection of $L^2_N$ onto the $NT$-periodic kernel of $\mathcal{A}_0[\phi]$ is given explicitly by[9]

$$\mathcal{P}_{0,N}f(x) = \frac{1}{N}\phi'(x)\left\langle\widetilde{\Phi}_0,f\right\rangle_{L^2_N}, \tag{3.4.5}$$

[8]Observe the $\xi=0$ term, corresponding to the projection onto the kernel of $\mathcal{A}_0[\phi]$ does not experience any type of temporal decay. Consequently, we factor it out of the remaining sums which, as we will see, do decay in time.

[9]Recall the left $T$-periodic eigenfunction $\widetilde{\Phi}_0$ is normalized so that $\left\langle\widetilde{\Phi}_0,\phi'\right\rangle_{L^2(0,T)}=1$, and hence $\left\langle\widetilde{\Phi}_0,\phi'\right\rangle_{L^2_N}=N$.

it follows from above that $S_{c,N}$ decomposes further as

$$\begin{aligned}
S_{c,N}f(x) &= e^{\mathcal{A}_0[\phi]t}\mathcal{P}_{0,N}f(x) + \frac{1}{NT}\sum_{\xi\in\Omega_N\setminus\{0\}}\rho(\xi)e^{i\xi x}e^{\lambda_c(\xi)t}\Phi_\xi(x)\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_T(f)(\xi,\cdot)\right\rangle_{L^2(0,T)} \\
&= e^{\mathcal{A}[\phi]t}\mathcal{P}_{0,N}f(x) + \phi'(x)\frac{1}{NT}\sum_{\xi\in\Omega_N\setminus\{0\}}\rho(\xi)e^{i\xi x}e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_T(f)(\xi,\cdot)\right\rangle_{L^2(0,T)} \\
&\quad + \frac{1}{NT}\sum_{\xi\in\Omega_N\setminus\{0\}}\rho(\xi)e^{i\xi x}i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_T(f)(\xi,\cdot)\right\rangle_{L^2(0,T)} \\
&=: e^{\mathcal{A}[\phi]t}\mathcal{P}_{0,N}f(x) + \phi'(x)s_{p,N}(t)f(x) + \widetilde{S}_{c,N}(t)f(x).
\end{aligned} \tag{3.4.6}$$

Now, arguing as in (3.3.8) we find that

$$\rho(\xi)\left|\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_T(f)(\xi,\cdot)\right\rangle_{L^2(0,T)}\right|^2 \lesssim \|f\|_{L^1_N}^2 \sup_{\xi\in[-\pi/T,\pi/T)}\left(\rho(\xi)\left\|\widetilde{\Phi}_\xi\right\|_{H^1_{\mathrm{per}}(0,T)}^2\right),$$

and hence, recalling the bound (3.3.6) and that

$$\left|\rho(\xi)^{1/2}e^{\lambda_c(\xi)t}\right| \lesssim e^{-d\xi^2 t}$$

for some constant $d > 0$, it follows by Parseval's identity (1.3.15) that

$$\begin{aligned}
\|\widetilde{S}_{c,N}(t)f\|_{L^2_N}^2 &= \frac{1}{NT^2}\sum_{\xi\in\Omega_N}\left\|\rho(\xi)i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_T(f)(\xi,\cdot)\right\rangle_{L^2(0,T)}\right\|_{L^2_x(0,T)}^2 \\
&\lesssim \left(\frac{1}{N}\sum_{\xi\in\Omega_N}\xi^2 e^{-2d\xi^2 t}\right)\|f\|_{L^1_N}^2.
\end{aligned} \tag{3.4.7}$$

Furthermore, following similar steps as above we have the bound

$$\left\|s_{p,N}(t)f\right\|_{L^2_N}^2 \lesssim \left(\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}}e^{-2d\xi^2 t}\right)\|f\|_{L^1_N}^2. \tag{3.4.8}$$

To complete the proof of Theorem 3.1.3, it remains to study the finite sums

$$\frac{1}{N}\sum_{\xi_j\in\Omega_N\setminus\{0\}} e^{-2d\xi_j^2 t} \;\text{ and }\; \frac{1}{N}\sum_{\xi_j\in\Omega_N} \xi_j^2 e^{-2d\xi_j^2 t}. \tag{3.4.9}$$

To obtain uniform (in $N$) rates of decay, note that both of the sums in (3.4.9) correspond to Riemann sum approximations to the integrals[10]

$$\int_{-\pi/T}^{\pi/T} e^{-2d\xi^2 t} d\xi \;\text{ and }\; \int_{-\pi/T}^{\pi/T} \xi^2 e^{-2d\xi^2 t} d\xi, \tag{3.4.10}$$

which, through elementary scaling analysis (as in the previous section), decay like $(1+t)^{-1/2}$ and $(1+t)^{-3/2}$, respectively. The next result uses this observation to obtain analogous decay bounds on the discrete sums in (3.4.9).

**Proposition 3.4.1.** *There exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $t > 0$ we have*

$$\frac{2\pi}{NT}\sum_{\xi_j\in\Omega_N\setminus\{0\}} e^{-2d\xi_j^2 t} \le C(1+t)^{-1/2} \tag{3.4.11}$$

*and*

$$\frac{2\pi}{NT}\sum_{\xi_j\in\Omega_N} \xi_j^2 e^{-2d\xi_j^2 t} \le C(1+t)^{-3/2}. \tag{3.4.12}$$

**Remark 3.4.2.** *The above bounds show that the polynomial decay rates on localized perturbations obtained in Theorem 3.1.5 provide uniform (in $N$) upper bounds on the decay rates associated to subharmonic perturbations, i.e. to perturbations with period $NT$ for some $N \in \mathbb{N}$. In the next section, we sharpen these estimates and provide associated lower bounds on the subharmonic decay rates, showing that, in fact, the localized decay rates provide* sharp *uniform bounds on subharmonic perturbations. The proof of the upper bounds in Proposition 3.4.1, however, is based on a substantially simpler monotonicity argument, which we now provide.*

*Proof of Proposition 3.4.1.* We compare the sums in (3.4.11) and (3.4.12) with the integrals in

[10] Recalling from Remark 1.3.4 that $\Delta\xi_j = 2\pi/NT$, we see that we technically need to multiply and divide by a harmless factor of $2\pi/T$.

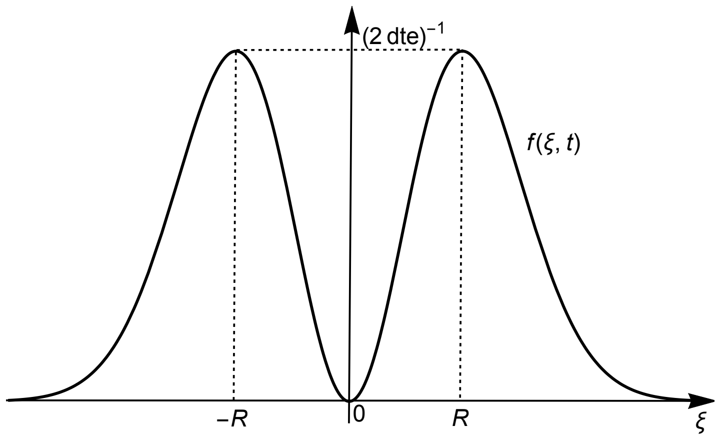



Figure 3.1: A schematic drawing of the function $\xi \mapsto \xi^2 e^{-2d\xi^2 t}$ used in the proof of Proposition 3.4.1. Note that the area under the supremum between $-R \le \xi \le R$ is $2R(2dte)^{-1} = 2e^{-1}(2dt)^{-3/2}$.

(3.4.10). For (3.4.11), notice that, for each $t > 0$, the function $\xi \mapsto e^{-2d\xi^2 t}$ is monotonically decreasing for $\xi > 0$ and monotonically increasing for $\xi < 0$. Together with the equality $(\xi_j - \xi_{j-1}) = 2\pi/NT$ these monotonicity property imply that

$$\begin{aligned}
\frac{2\pi}{NT} \sum_{\xi_j \in \Omega_N \setminus \{0\}} e^{-2d\xi_j^2 t} &= \sum_{\substack{\xi_j \in \Omega_N \\ \xi_j < 0}} e^{-2d\xi_j^2 t} (\xi_{j+1} - \xi_j) + \sum_{\substack{\xi_j \in \Omega_N \\ \xi_j > 0}} e^{-2d\xi_j^2 t} (\xi_j - \xi_{j-1}) \\
&\le \sum_{\substack{\xi_j \in \Omega_N \\ \xi_j < 0}} \int_{\xi_j}^{\xi_{j+1}} e^{-2d\xi^2 t} d\xi + \sum_{\substack{\xi_j \in \Omega_N \\ \xi_j > 0}} \int_{\xi_{j-1}}^{\xi_j} e^{-2d\xi^2 t} d\xi \qquad (3.4.13) \\
&\le \int_{-\pi/T}^{\pi/T} e^{-2d\xi^2 t} d\xi,
\end{aligned}$$

which proves the inequality (3.4.11).

For (3.4.12), we have to slightly modify this argument because the function $\xi \mapsto \xi^2 e^{-2d\xi^2 t}$ does not have the same monotonicity properties. Indeed, it has a global minimum at $\xi = 0$ and two global maxima at $\xi = \pm R$ where

$$R = (2dt)^{-1/2} \quad \text{and} \quad R^2 e^{-2dR^2 t} = (2dte)^{-1};$$

see Figure 3.1. Then for $0 < t \le T^2/2d\pi^2$, the values $\pm R$ do not belong to the interval $(-\pi/T, \pi/T)$

and we can easily estimate the sum in (3.4.12),

$$\frac{2\pi}{NT}\sum_{\xi_j\in\Omega_N}\xi_j^2 e^{-2d\xi_j^2 t} \le \frac{2\pi}{NT}\sum_{\xi_j\in\Omega_N}\frac{\pi^2}{T^2} \le \frac{2\pi^3}{T^3}. \tag{3.4.14}$$

For $t > T^2/2d\pi^2$, we consider the function

$$G_t(\xi) := \begin{cases} (2dte)^{-1}, & |\xi| \le R \\ \xi^2 e^{-2d\xi^2 t}, & R < |\xi| \le \pi/T \end{cases}$$

which is nonincreasing for $\xi > 0$ and nondecreasing for $\xi < 0$. Then by arguing as for (3.4.13), we find

$$\begin{aligned}
\frac{2\pi}{NT}\sum_{\xi_j\in\Omega_N\setminus\{0\}}\xi_j^2 e^{-2d\xi_j^2 t} &= \sum_{\substack{\xi_j\in\Omega_N\\ \xi_j<0}}\xi_j^2 e^{-2d\xi_j^2 t}(\xi_{j+1}-\xi_j) + \sum_{\substack{\xi_j\in\Omega_N\\ \xi_j>0}}\xi_j^2 e^{-2d\xi_j^2 t}(\xi_j-\xi_{j-1}) \\
&\le \sum_{\substack{\xi_j\in\Omega_N\\ \xi_j<0}}\int_{\xi_j}^{\xi_{j+1}} G_t(\xi)d\xi + \sum_{\substack{\xi_j\in\Omega_N\\ \xi_j>0}}\int_{\xi_{j-1}}^{\xi_j} G_t(\xi)d\xi \\
&\le \int_{-\pi/T}^{\pi/T} G_t(\xi)d\xi \;\le\; 2e^{-1}(2dt)^{-3/2} + \int_{-\pi/T}^{\pi/T}\xi^2 e^{-2d\xi^2 t}d\xi.
\end{aligned}$$

Together with (3.4.14) this proves the inequality (3.4.12) and completes the proof of the proposition.

□

In summary, for each $f \in L^2_N$ we have the decomposition

$$\begin{aligned}
e^{\mathcal{A}[\phi]t} f(x) &= e^{\mathcal{A}[\phi]t}\mathcal{P}_{0,N} f(x) + \phi'(x)s_{p,N}(t)f(x) \\
&\quad + \widetilde{S}_{c,N}(t)f(x) + \widetilde{S}_{lf,N}(t)f(x) + S_{hf,N}(t)f(x),
\end{aligned}$$

where $\mathcal{P}_{0,N}$ is the projection of $L^2_N$ onto the $NT$-periodic kernel of $\mathcal{A}[\phi]$ defined in (3.4.5), the operators $s_{p,N}(t)$ and $\widetilde{S}_{lf,N}(t)$ are defined as in (3.4.6), and the operators $\widetilde{S}_{c,N}(t)$ and $S_{hf,N}(t)$ are defined as in (3.4.3) and (3.4.1), respectively. Recalling the estimates (3.4.7)-(3.4.8), Proposition

3.4.1 implies that for all $N \in \mathbb{N}$ we have

$$\left\|s_{p,N}(t)f\right\|_{L^2_N} \lesssim (1+t)^{-1/4}\|f\|_{L^1_N}, \quad \text{and} \quad \left\|\widetilde{S}_{c,N}(t)f\right\|_{L^2_N} \lesssim (1+t)^{-3/4}\|f\|_{L^1_N},$$

valid for all $t \geq 0$. Together with the exponential decay estimates (3.4.4) and (3.4.2) on $\widetilde{S}_{lf,N}(t)$ and $S_{hf,N}(t)$, respectively, and defining for each $f \in L^2_N$ the function[11]

$$\gamma_N(x,t) := \frac{1}{N}\left\langle \widetilde{\Phi}_0, f\right\rangle_{L^2_N} + s_{p,N}(t)f(x)$$

and noting that

$$\frac{1}{N}\left\langle \widetilde{\Phi}_0, f\right\rangle_{L^2_N} = \frac{\left\langle \phi', \mathcal{P}_{0,N}f\right\rangle_{L^2(0,T)}}{\|\phi'\|^2_{L^2(0,T)}}$$

this completes the proof of Theorem 3.1.3.

**Remark 3.4.3.** *As mentioned near the end of the introduction of this chapter, the methodology used in the above proof is very general and applies more generally to linear operators $\mathcal{A}[\phi]$ with $T$-periodic coefficients that satisfy conditions (i)-(iii) listed in the discussion after Theorem 3.1.5. The resolvent bound in (iii) implies exponential decay of high-frequency modes, while the diffusive spectral stability condition yields polynomial decay rates on the critical modes corresponding to spectrum near $(\lambda, \xi) = (0,0)$. This work thus sets forth a general methodology for establishing uniform decay rates on subharmonic perturbations of diffusively spectrally stable periodic waves in a large class of evolution equations.*

Finally, we note that exponential decay rates on the semigroup $e^{\mathcal{A}[\phi]t}$ acting on $L^2_N$ may also be obtained from the above analysis. Indeed, note that for each fixed $N \geq 2$ we have the bounds[12]

$$\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} e^{-2d\xi^2 t} \leq \left(1-\frac{1}{N}\right)e^{-2d(\Delta\xi)^2 t}$$

[11] Clearly $\gamma_N$ is an $NT$-periodic function of $x$.

[12] Note that since $\Omega_1 = \{0\}$, in the case $N = 1$ we clearly have $\sum_{\xi\in\Omega_1\setminus\{0\}} e^{-2d\xi^2 t} = 0$ and $\sum_{\xi\in\Omega_1} \xi^2 e^{-2d\xi^2 t} = 0$ for all $t \geq 0$.

and, similarly,

$$\frac{1}{N}\sum_{\xi\in\Omega_N}\xi^2 e^{-2d\xi^2 t} \le \left(1-\frac{1}{N}\right)\left(\frac{\pi}{T}\right)^2 e^{-2d(\Delta\xi)^2 t},$$

where here[13]

$$\Delta\xi = \frac{2\pi}{NT}.$$

In particular, from (3.4.6)-(3.4.8) and the decompositions (3.4.1)-(3.4.4) it immediately follows that there exists a constant $C > 0$ such that for each $N \in \mathbb{N}$ and $f \in L^2_N$ we have the exponential decay bound[14]

$$\left\| e^{\mathcal{A}[\phi]t}\left(1-\mathcal{P}_{0,N}\right) f\right\|_{L^2_N} \le Ce^{-\delta t}\|f\|_{L^1_N\cap L^2_N}, \qquad \delta := \min\{\eta, d(\Delta\xi)^2\}, \tag{3.4.15}$$

with $\eta > 0$ as in (3.4.4), recovering, at the linear level, the exponential stability result from [67]: see Theorem 3.1.2 in the introduction to this chapter. In fact, the above observation extends the exponential bound result used in [67] since the constant $C > 0$ above does not depend on $N$: see also Remark 3.2.6(ii) where the constant depends on $N$. Observe, however, that the exponential rate of decay exhibited above still tends to zero as $N \to \infty$.

## 3.5 Sharpness of Localized Theory

While the polynomial decay rates established in Proposition 3.4.1 provide upper bounds on the uniform decay rates of subharmonic perturbations of a given diffusively spectrally stable, $T$-periodic standing wave solution of the LLE (3.1.1), it is not a-priori clear that such decay rates are sharp, i.e., if it is possible that subharmonic perturbations actually experience even faster uniform rates of decay. After all, for each fixed $N \in \mathbb{N}$ we already know such perturbations exhibit exponential decay (up to a null translational mode). The next result shows that, in fact, the localized decay rates provided in Theorem 3.1.5 provide *sharp uniform decay rates* for subharmonic perturbations.

[13]Note for large $N$ that the size of the spectral gap $\max_{\xi\in\Omega_N\setminus\{0\}}\Re(\lambda_c(\xi))$ is $O((\Delta\xi)^2)$.
[14]In fact, we can improve (3.4.15) so that it is a bound from $L^2_N \to L^2_N$.

**Proposition 3.5.1.** *There exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $t > 0$ we have*

$$\left| \frac{2\pi}{NT} \sum_{\xi_j \in \Omega_N \setminus \{0\}} e^{-2d\xi_j^2 t} - \int_{-\pi/T}^{\pi/T} e^{-2d\xi^2 t} d\xi \right| \leq \frac{C}{N} \tag{3.5.1}$$

*and*

$$\left| \frac{2\pi}{NT} \sum_{\xi_j \in \Omega_N} \xi_j^2 e^{-2d\xi_j^2 t} - \int_{-\pi/T}^{\pi/T} \xi^2 e^{-2d\xi^2 t} d\xi \right| \leq \frac{C}{N(1+t)}. \tag{3.5.2}$$

**Remark 3.5.2.** *Note that since the terms $e^{-2d\xi^2 t}$ are exponentially small outside a small ball near $\xi = 0$ for large enough time, the domain of integration above could be replaced with $(-\infty, \infty)$, giving an even more direct connection to the bounds* (3.4.7)-(3.4.8) *in the subharmonic case and the bounds* (3.3.10) *and* (3.3.9) *in the localized case.*

The proof of Proposition 3.5.1 is based on a careful rescaling and Riemann sum argument and is included in Appendix 3.A. As a consequence we see convergence, say in $L_t^\infty(0,\infty)$, as $N \to \infty$ of the discrete sum to the integral associated to the localized theory. This proves that the *uniform* (in $N$) bounds on the operators $s_{p,N}(t)$ and $\widetilde{S}_{c,N}(t)$ provided in Theorem 3.1.3 are *sharp*. Of course, as mentioned above and several times in the manuscript, while each subharmonic perturbation exhibits exponential decay in time, it follows that the localized theory precisely describes subharmonic decay rates which are uniform in $N$. Furthermore, as shown in the proof, the estimates (3.5.1)-(3.5.2) show explicitly that for a fixed $N$ the sums are good approximations of the associated integral on time scales at most[15] $t = O(N^2)$. For even larger times, the exponential nature of the summands dominate and the sum decays monotonically to zero at an exponential rate.

Naturally, it is interesting to try to recover the localized theory from the subharmonic theory in the limit as $N \to \infty$. For example, using the boundedness of $\widetilde{\Phi}_0$ we note there exists a constant

[15]This behavior on times scales at most $O(N^2)$, as well as the exponential behavior for larger times, is readily observed numerically.

$C > 0$ independent of $N \in \mathbb{N}$ such that[16]

$$\left| \left\langle \widetilde{\Phi}_0, f \right\rangle_{L^2_N} \right| \leq C \|f\|_{L^1_N}. \tag{3.5.3}$$

and hence the triangle inequality implies that

$$\left\| e^{\mathcal{A}[\phi]t} f \right\|_{L^2_N} \leq C \left( \frac{1}{\sqrt{N}} + (1+t)^{-1/4} \right) \|f\|_{L^1_N \cap L^2_N}.$$

Consequently, if $\{f_N\}_{N=1}^{\infty}$ is a sequence of functions with $f_N \in L^2_N \cap L^1_N$ for each $N$ and if there exists a $v \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f_N \to v$, say in $L^1_{\text{loc}}(\mathbb{R})$, then formally taking $N \to \infty$ allows us to (again formally) recover the stability result to the localized perturbation $v$ established in Theorem 3.1.5. Furthermore, the estimate (3.5.3) suggests that we should have (in some appropriate sense)

$$\lim_{N \to \infty} \gamma_N(x, t; f_N) = \gamma(x, t; v),$$

where $\gamma$ is the modulation function associated to $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$; that is, we should have convergence of the associated subharmonic and localized modulation functions. Of course, to make rigorous sense of these (and other) limiting results one must deal with the fact that a sequence of $NT$-periodic functions can only converge to a function in $L^2(\mathbb{R})$ at best locally in space. Establishing such a convergence results rigorously is left as an open problem.

## 3.6 Towards Nonlinear Stability

Our main results, Theorem 3.1.3, Theorem 3.1.5, and Corollary 3.3.2 give insight into the asymptotic stability and long-time modulational dynamics near a diffusively spectrally stable periodic stationary solution of the Lugiato-Lefever equation (3.1.1) to both subharmonic and localized perturbations. Specifically, we note all these results are established at the linear level. Our ultimate

[16]Note if we use the Cauchy-Schwartz inequality to control the quantity (3.5.3) by $L^2_N$, decay in $N$ is not observed due to the $T$-periodicity of $\widetilde{\Phi}_0$.

goal, however, is to extend these results to the nonlinear level. In this section, we describe an approach to this problem which has been useful in the related context of dissipative conservation or balance laws [31], as well as the difficulties that must be overcome for the LLE.

To begin, suppose $\phi$ is a $T$-periodic, diffusively spectrally stable stationary periodic solution of (3.1.1) and consider (3.1.1) equipped with the initial condition

$$u(x,0) = \phi(x) + v_0(x) \tag{3.6.1}$$

for some sufficiently smooth and small initial perturbation $v_0$. For the sake of clarity, let us assume the initial perturbation is localized, i.e., that $v_0 \in L^2(\mathbb{R})$. So long as it exists, Theorem 3.1.5 suggests decomposing the solution $u(x,t)$ of (3.6.1) as

$$u(x,t) = \phi(x + \gamma(x,t)) + v(x,t)$$

where here $\gamma$ is some appropriate spatial modulation to be specified[17] as needed in the analysis. Substituting this ansatz into (3.6.1) implies the perturbation $v$ satisfies an evolution equation of the form

$$(\partial_t - \mathcal{A}[\phi])\,(v - \gamma\phi') = \mathcal{N}(v, v_x, v_{xx}, \gamma_t, \gamma_x, \gamma_{xx}), \tag{3.6.2}$$

where $\mathcal{N}$ consists of nonlinear terms in its arguments and their derivatives. Using Duhamel, the above nonlinear evolution equation is equivalent to the following (implicit) integral equation

$$v(x,t) - \gamma(x,t)\phi'(x) = e^{\mathcal{A}[\phi]t}v(x,0) + \int_0^t e^{\mathcal{A}[\phi](t-s)}\mathcal{N}(v, v_x, v_{xx}, \gamma_t, \gamma_x, \gamma_{xx})(x,s)ds. \tag{3.6.3}$$

Since the linear estimates in Theorem 3.1.5 imply the linearized solution operator can be decomposed as

$$e^{\mathcal{A}[\phi]t}f(x) = \phi'(x)\left(s_p(t)f\right)(x) + \widetilde{S}(t)f(x)$$

[17] Thanks to the invariance of (3.1.1) with respect to spatial translations, we may assume that $\gamma(x,0) = 0$.

where $s_p$ is defined as in (3.3.7) and $\|\widetilde{S}(t)f\|_{L^2(\mathbb{R})} \lesssim (1+t)^{-3/4}\|f\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}$, it is natural to choose the modulation $\gamma$ above to exactly cancel all the $s_p$ contributions on the right hand side of (3.6.3), thus leaving a coupled set of integral equations for $v$ and $\gamma$ that one might hope to solve and study via contraction.

At this point in the argument, things begin to break down. In particular, observe that control over the source term in (3.6.3) in $L^2(\mathbb{R})$ would require control over (at least) $v$ in $H^2(\mathbb{R})$, corresponding to a loss of derivatives in $v$ and a-priori leaving little hope of studying (3.6.3) via iteration. In some cases, however, such a loss of derivatives at the linear level can be compensated by appropriately strong damping effects of the governing evolution equation: see, for example, [31]. However, in the case of the LLE (3.1.1) the damping actually corresponds to the *lowest-order derivative* [18] which elementary calculations show negates the "nonlinear damping" technique leveraged in [31] to regain derivatives at the nonlinear level. More explicitly, this can be seen in the following remark.

**Remark 3.6.1.** *Note that for the classical linear heat equation*

$$u_t = u_{xx}$$

*posed on $L^2(\mathbb{R})$ an elementary calculation (multiplying the linear heat equation by $u - u_{xx}$ and integrating over $\mathbb{R}$), shows that*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^1}^2 = -\|u_x\|_{H^1}^2 \leq -\|u(t)\|_{H^1}^2 + \|u(t)\|_{L^2}^2.$$

*By Gronwall's inequality, this in turn implies that*

$$\|u(t)\|_{H^1}^2 \leq e^{-2t}\|u(0)\|_{H^1}^2 + 2\int_0^t e^{-2(t-s)}\|u(s)\|_{L^2}^2 ds,$$

*showing that the $H^1$-norm of solutions, for so long as they exist, are slaved to the $L^2$-norm. The key to the above calculuation is that the damping occurred (trivially) in the* highest order derivative

[18] In fact, a zeroth order term.

*of the equation. In the case of the LLE equation* (5.1.1)*, however, the damping more resembles the linear equation*

$$w_t = -w$$

*which, by similar calculations as above, implies solutions satisfy the estimate*

$$\|w(t)\|_{H^1}^2 = e^{-2t}\|w(0)\|_{H^1}^2,$$

*which does not yield the desired control over* $H^1$*. Consequently, the low order damping of the LLE equation* (5.1.1) *precludes the use of nonlinear damping estimates in the present case.*

In summary, when attempting to upgrade our linear results (either in the localized or subharmonic setting) to a nonlinear theory one is confronted with an iteration scheme which a-priori loses regularity. However, we know that in cases where the dissipation is sufficiently strong, we can use the nonlinear damping technique outlined in Remark 3.6.1 to regain this lost regularity, at least in the case of localized perturbations. Consequently, this motivates the structure for the rest of the dissertation. Recalling that the localized case served as motivation for the uniform study of subharmonic perturbations, we first seek to understand if it is possible – in the presence of sufficiently strong damping, allowing one to circumvent the loss of regularity via a nonlinear damping result – to establish a nonlinear stability result to subharmonic perturbations that is uniform in the period of the perturbations, which is the focus of Chapter 4. Once we confirm that this is, in fact, possible, we turn our attention back to the LLE in Chapter 5 and seek to develop a new methodology that allows us to circumvent the loss of regularity in the case of localized perturbations and in the presence of weak damping, i.e., in the absence of a nonlinear damping estimate. With this methodology in hand, along with the intuition provided from Chapter 4, we are currently studying the ability to use this to establish the desired uniform subharmonic stability result for the Lugiato-Lefever equation [23].

# Appendix

## 3.A Proof of Proposition 3.5.1

In the appendix to this chapter, we present a proof of Proposition 3.5.1, which establishes that the localized rates of decay in Theorem 3.1.5 provide *sharp* uniform bounds on subharmonic perturbations of diffusively spectrally stable periodic standing waves of the LLE (3.1.1).

*Proof of Proposition 3.5.1.* As in the proof of Proposition 3.4.1, it is sufficient to establish the bounds (3.5.1)-(3.5.2) for $N \geq 2$ and $t \geq 1$. Since the proof of the estimates (3.5.1) and (3.5.2) for $N \geq 2$ and $t \geq 1$ follow the same basic structure, we present a detailed proof of (3.5.2) and then describe the modifications needed to establish (3.5.1).

To begin our proof of (3.5.2), fix $N \in \mathbb{N}$ with $N \geq 2$ and define the function

$$F_N(t) := t^{3/2} \sum_{\xi_j \in \Omega_N} \xi_j^2 e^{-2d\xi_j^2 t} \Delta\xi_j, \quad t \geq 1, \tag{3.A.1}$$

where here for each $j \in \mathbb{N}$ we set $\Delta\xi_j = \frac{2\pi}{NT}$. We now fix $t > 0$ and use a discrete change of variables by setting, for each $\xi_j \in \Omega_N$,

$$z_j = \xi_j \sqrt{t} \tag{3.A.2}$$

noting that, in particular, the set $\Omega_{N,t} := \{z_j\}$ is a discretization of $[-\pi\sqrt{t}/T, \pi\sqrt{t}/T]$ with

$$\Delta z_j = \Delta\xi_j \sqrt{t} = 2\pi\sqrt{t}/(NT).$$

Further, in terms of the new variables we have

$$F_N(t) = \sum_{z_j \in \Omega_{N,t}} z_j^2 e^{-2dz_j^2} \Delta z_j,$$

and hence $F_N$ is a Riemann sum approximation for the integral

$$\int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} H(z)dz, \quad \text{where} \quad H(z) := z^2 e^{-2dz^2}.$$

Our strategy is to treat $F_N$ as either a left or right endpoint Riemann sum approximation of the integral, and our choice will be dictated by the intervals where $H$ is increasing or decreasing. From this, we will establish the bound[19]

$$\left| F_N(t) - \int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} H(z)dz \right| \le \frac{C\sqrt{t}}{N}, \tag{3.A.3}$$

which (by undoing the above rescaling), is clearly equivalent to the bound (3.5.2).

Now, to establish (3.A.3) we first note by symmetry of the function $H$ that it suffices to study the sum defining $F_N$ over only the non-negative $z_j$, i.e. to study the function

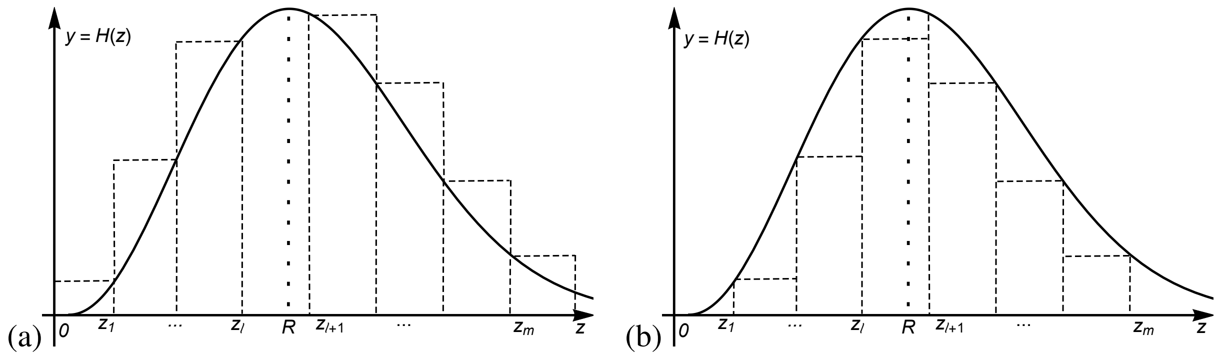
$$h_N(t) := \sum_{j=0}^{m} z_j^2 e^{-2dz_j^2} \Delta z_j = \sum_{j=0}^{m} H(z_j)\Delta z_j, \quad m := \max\{j \mid z_j \in \Omega_{N,t}\} = \begin{cases} \frac{N}{2} - 1, & N \text{ even} \\ \frac{N-1}{2}, & N \text{ odd} \end{cases}$$

Observe that $H$ is monotonically increasing for $z \in (0, R)$, where $R = (2d)^{-1/2}$, and is monotonically decreasing for $z \in (R, \infty)$. In particular, for each fixed $t > 0$, we have that either $z_m = z_m(t) < R$ or there exists $\ell = \ell(t) \in \{0, \dots, m-1\}$ such that $R \in (z_\ell, z_{\ell+1}]$: see Figure 3.A.1.

Note that if $\ell = 0$ then $z_j > R$ for each $j \in \{1, 2, \dots, m\}$, implying that each such $z_j$ lies in the monotonically decreasing tail of $H$. This case happens when $z_1 > R$, i.e. when $t > C_0 N^2$ where $C_0 = T^2/(8d\pi^2)$. Noting, as in Figure 3.A.1, that the $z_\ell$ and $z_{\ell+1}$ terms need to be handled differently, we decompose $h_N$ as

$$h_N(t) = \sum_{j=0}^{1} H(z_j)\Delta z_j + \sum_{j=2}^{m} H(z_j)\Delta z_j.$$

[19]Note this clearly shows that the finite sum is a good approximation of the associated integral on time scales at most $t = O(N^2)$.




(a) (b)

Figure 3.A.1: (a) Treating $h_N$ as an over approximation of the area under $H$. (b) Treating $h_N$ as an under approximation. Notice that (a) misses the interval $[z_\ell, z_{\ell+1}]$. Similarly (b) has a node, in this case $z_{\ell+1}$, that cannot produce an under approximation on the interval from $[z_\ell, z_{\ell+1}]$. This indicates that this interval must be treated with some care.

On the opposite of this extreme, observe that if $\ell = m-1$ then $z_j < R$ for all $j \in \{0, 1, \ldots, m-1\}$, implying that each such $z_j$ lies in the monotonically increasing portion of $H$, while $z_m \geq R$, leading to a similar decomposition to that above. Further, we only treat $z_m$ differently when $z_m < R$, which occurs for some (fixed) bounded interval of time. (Although, we note that, strictly speaking, $z_m$ does not have to be treated differently when $z_m + \Delta z_m < R$.)

In our analysis below, we will only consider the case when there are some $z_j$ in both the monotonically increasing and the monotonically decreasing components of $H$. That is, we restrict to the case when

$$R \in (z_\ell, z_{\ell+1}], \quad \text{for some } \ \ell \in \{1, \ldots, m-2\}, \tag{3.A.4}$$

noting the other more extreme cases described above will follow in a more straightforward way. In the case when (3.A.4) holds, we decompose $h_N$ into thee components via

$$h_N(t) = \sum_{j=0}^{\ell-1} H(z_j)\Delta z_j + \sum_{j=\ell}^{\ell+1} H(z_j)\Delta z_j + \sum_{j=\ell+2}^{m} H(z_j)\Delta z_j.$$

Note the first and last terms represent the sums over the $j$ where $z_{j+1} < R$ and where $z_{j-1} > R$, respectively, while the middle term represents the sum over the $z_j$ nearest to the absolute maximum of $H$.

Because $H$ is increasing on $(0, R)$ it follows that

$$H(z_j)\Delta z_j \leq \int_{z_j}^{z_{j+1}} H(z)dz \leq H(z_{j+1})\Delta z_{j+1} \text{ for } j = 0, 1, \ldots, \ell - 1,$$

and hence, recalling that $H(z_0) = 0$, we have[20]

$$\int_0^{z_{\ell-1}} H(z)dz \leq \sum_{j=0}^{\ell-1} H(z_j)\Delta z_j \leq \int_0^{z_\ell} H(z)dz.$$

Similarly, since $H$ is decreasing on $(R, \infty)$ it follows that

$$H(z_j)\Delta z_j \leq \int_{z_{j-1}}^{z_j} H(z)dz \leq H(z_{j-1})\Delta z_{j-1} \text{ for } j = \ell + 2, \ldots, m$$

which, as above, yields

$$\int_{z_{\ell+2}}^{z_m+\Delta z_m} H(z)dz \leq \sum_{j=\ell+2}^{m} H(z_j)\Delta z_j \leq \int_{z_{\ell+1}}^{z_m} H(z)dz.$$

Together, this gives

$$\int_0^{z_m+\Delta z_m} H(z)dz - \int_{z_{\ell-1}}^{z_{\ell+2}} H(z)dz \leq h_N(t) - \sum_{j=\ell}^{\ell+1} H(z_j)\Delta z_j \leq \int_0^{z_m} H(z)dz - \int_{z_\ell}^{z_{\ell+1}} H(z)dz. \tag{3.A.5}$$

It now remains to bound the sum $\sum_{j=\ell}^{\ell+1} H(z_j)\Delta z_j$. To this end, recall that $H$ has exactly one critical point (a global maximum at $z = R$) on $(z_\ell, z_{\ell+1}]$. Consequently, since the minimum of $H$ on $[z_\ell, z_{\ell+1}]$ must occur at one of the endpoints, we have

$$\min_{j=\ell,\ell+1} H(z_j)\Delta z_j \leq \int_{z_\ell}^{z_{\ell+1}} H(z)dz.$$

[20]Technically, the above bound gives $\int_0^{z_{\ell-1}} H(z)dz \leq \sum_{j=1}^{\ell-1} H(z_j)\Delta z_j$,. Since $H(z_0) = 0$, however, this is equivalent to the stated lower estimate.

while the other endpoint is at worst the supremum of $H$, yielding

$$\max_{j=\ell,\ell+1} H(z_j)\Delta z_j \leq H(R)\left(2\pi\sqrt{t}/(NT)\right) = C\sqrt{t}/N,$$

where $C = \pi/(dTe)$. The previous two bounds together yield

$$\sum_{j=\ell}^{\ell+1} H(z_j)\Delta z_j \leq \int_{z_\ell}^{z_{\ell+1}} H(z)dz + C\sqrt{t}/N.$$

On the other hand, recalling that $H$ is increasing on $(z_{\ell-1}, z_\ell)$ and is decreasing on $(z_{\ell+1}, z_{\ell+2})$ gives the lower bound

$$H(z_\ell)\Delta z_\ell \geq \int_{z_{\ell-1}}^{z_\ell} H(z)dz, \quad \text{and} \quad H(z_{\ell+1})\Delta z_{\ell+1} \geq \int_{z_{\ell+1}}^{z_{\ell+2}} H(z)dz.$$

Together with the estimate (3.A.5), it follows that

$$\int_0^{z_m+\Delta z_m} H(z)dz - \int_{z_\ell}^{z_{\ell+1}} H(z)dz \leq h_N(t) \leq \int_0^{z_m} H(z)dz + C\sqrt{t}/N. \tag{3.A.6}$$

Recalling again that $H$ attains its global maximum at $z = R \in [z_\ell, z_{\ell+1}]$, we clearly have

$$\int_{z_\ell}^{z_{\ell+1}} H(z)dz \leq H(R)\left(2\pi\sqrt{t}/(NT)\right) = C\sqrt{t}/N.$$

and hence, from (3.A.6), we obtain

$$\int_0^{z_m+\Delta z_m} H(z)dz - C\sqrt{t}/N \leq h_N(t) \leq \int_0^{z_m} H(z)dz + C\sqrt{t}/N. \tag{3.A.7}$$

By the even symmetry of $H$ and the structure of the sets $\Omega_N$, we find that

$$\int_{z_{-\widetilde{m}}-\Delta z_{-\widetilde{m}}}^{0} H(z)dz - C\sqrt{t}/N \leq \sum_{j=-\widetilde{m}}^{-1} H(z_j)\Delta z_j \leq \int_{z_{-\widetilde{m}}}^{0} H(z)dz + C\sqrt{t}/N, \tag{3.A.8}$$

where

$$\widetilde{m} := |\min\{j \mid z_j \in \Omega_{N,t}\}| = \begin{cases} \frac{N}{2}, & N \text{ even} \\ \frac{N-1}{2}, & N \text{ odd} \end{cases}.$$

Therefore, combining (3.A.7) and (3.A.8) we obtain the bound

$$\int_{z_{-\widetilde{m}}-\Delta z_{-\widetilde{m}}}^{z_m+\Delta z_m} H(z)dz - 2C\sqrt{t}/N \leq F_N(t) \leq \int_{z_{-\widetilde{m}}}^{z_m} H(z)dz + 2C\sqrt{t}/N.$$

Noting that

$$\int_{z_{-\widetilde{m}}}^{z_m} H(z)dz \leq \int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} H(z)dz \leq \int_{z_{-\widetilde{m}}-\Delta z_{-\widetilde{m}}}^{z_m+\Delta z_m} H(z)dz,$$

we obtain the bound (3.A.3), as desired. By undoing the rescaling (3.A.2), this establishes the estimate (3.5.2) in Proposition 3.5.1.

Finally, the proof of the estimate (3.5.1) follows by a similar, yet simpler, analysis. Indeed, for all $N \geq 2$ and $t \geq 1$ we define

$$\widetilde{F}_N(t) := t^{1/2} \sum_{\xi_j \in \Omega_N \setminus \{0\}} e^{-2d\xi_j^2 t} \Delta\xi_j$$

and note by performing the same discrete change of variables in (3.A.2) as before we have

$$\widetilde{F}_N(t) = \sum_{z_j \in \Omega_{N,t} \setminus \{0\}} e^{-2dz_j^2} \Delta z_j.$$

Since the function $\widetilde{H}(z) = e^{-2dz^2}$ is even and strictly decreasing for $z > 0$, carrying out the same monotonicity argument as above we find that

$$\int_{z_{-\widetilde{m}}-\Delta z_{-\widetilde{m}}}^{z_m+\Delta z_m} \widetilde{H}(z)dz - \int_{z_{-1}}^{z_1} \widetilde{H}(z)dz \leq \widetilde{F}_N(t) \leq \int_{z_{-\widetilde{m}}}^{z_m} \widetilde{H}(z)dz.$$

In particular, using the elementary bound

$$\int_{z_{-1}}^{z_1} \widetilde{H}(z)dz \leq \widetilde{H}(0)(z_1 - z_{-1}) = \frac{4\pi\sqrt{t}}{NT}$$

and noting that

$$\int_{z_{-\widetilde{m}}}^{z_m} \widetilde{H}(z)dz \leq \int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} \widetilde{H}(z)dz \leq \int_{z_{-\widetilde{m}}-\Delta z_{-\widetilde{m}}}^{z_m+\Delta z_m} \widetilde{H}(z)dz,$$

we obtain the estimate

$$\left|\widetilde{F}_N(t) - \int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} e^{-2dz^2}dz\right| \leq \frac{4\pi\sqrt{t}}{NT} \tag{3.A.9}$$

valid for all $N \geq 2$ and $t \geq 1$. Undoing the rescaling (3.A.2), this completes the proof of (3.4.11). □

**Remark 3.A.1.** *The reader may wonder why we used a rescaling and monotonicity argument above to establish* (3.A.3) *and* (3.A.9)*, as opposed to a more direct Mean Value Theorem argument. Essentially, this is because the Mean Value Theorem gives us the cruder bound*

$$\left|F_N(t) - \int_{-\pi\sqrt{t}/T}^{\pi\sqrt{t}/T} H(z)dz\right| \leq \frac{Mt}{N}.$$

Finally we note that the uniform bounds in Proposition 3.4.1 may be recovered from the above analysis. For example, note that the function $F_N$ defined in (3.A.1) satisfies the differential inequality

$$F_N'(t) \leq t^{1/2}\left(\frac{3}{2} - \frac{8d\pi^2 t}{N^2T^2}\right)\sum_{\xi_j\in\Omega_N} \xi_j^2 e^{-2d\xi_j^2 t}\Delta\xi_j = t^{-1}\left(\frac{3}{2} - \frac{8d\pi^2 t}{N^2T^2}\right)F_N(t)$$

and hence exhibits exponential decay to zero for time scales larger than $O(N^2)$. Since (3.A.3) shows that $F_N$ is uniformly bounded on time scales up to $O(N^2)$ it follows that $F_N(t)$ is uniformly bounded (in $N$) for all $t \geq 1$, which establishes (3.4.12) in Proposition 3.4.1. A similar differential inequality argument applied to $\widetilde{F}_N(t)$ establishes the uniform estimate (3.4.11).

# Chapter 4

# Nonlinear Subharmonic Dynamics of Spectrally Stable Periodic Waves in Reaction Diffusion Systems


## Abstract

In this chapter, we investigate the stability and nonlinear local dynamics of spectrally stable wave trains in reaction-diffusion systems. For each $N \in \mathbb{N}$, such $T$-periodic traveling waves are easily seen to be nonlinearly asymptotically stable (with asymptotic phase) with exponential rates of decay when subject to $NT$-periodic, i.e., subharmonic, perturbations. However, both the allowable size of perturbations and the exponential rates of decay depend on $N$, and, in particular, they tend to zero as $N \to \infty$, leading to a lack of uniformity in such subharmonic stability results. In this work, we build on recent work by the authors and introduce a methodology that allows us to achieve a stability result for subharmonic perturbations which is uniform in $N$. Our work is motivated by the dynamics of such waves when subject to perturbations which are localized (i.e. integrable on the line), which has recently received considerable attention by many authors.

In the broader goal of this thesis, this chapter serves as a proof of concept. Namely, we aim to verify if the presence of sufficiently strong damping, i.e., if the presence of a "nonlinear damping" estimate, allows us to upgrade the linear uniform subharmonic stability result attained in the previous chapter to an appropriate nonlinear stability result. This chapter is representative of [33], which was a joint work with Mathew Johnson.

## 4.1 Introduction

In this work, we consider the local dynamics of periodic traveling wave solutions, i.e., wave trains, in reaction diffusion systems of the form

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \quad t \geq 0, \quad u \in \mathbb{R}^n \tag{4.1.1}$$

where $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^K$-smooth nonlinearity for some $K \geq 3$. Such systems arise naturally in many areas of applied mathematics, and the behavior of such wave train solutions when subject to a variety of classes of perturbations has been studied intensively over the last decade. Most commonly in the literature, one studies the stability and instability of such periodic traveling waves to perturbations which are *localized*, i.e. integrable on the line, or which are *nonlocalized*, accounting for asymptotic phase differences at infinity. See, for example, [13, 29, 30, 37, 56, 57] and references therein.

Here, we consider the stability and long-time dynamics of $T$-periodic traveling wave solutions of (4.1.1) when subjected to $NT$-periodic, i.e. *subharmonic*, perturbations for some $N \in \mathbb{N}$. More precisely, suppose that $u(x,t) = \phi(k(x - ct))$ is a periodic traveling wave solution of (4.1.1) with period $T = 1/k$, where we choose $k \in \mathbb{R}$ so that the profile $\phi \in H^1_{\text{loc}}(\mathbb{R})$ is a 1-periodic stationary solution of

$$ku_t - kcu_x = k^2 u_{xx} + f(u), \tag{4.1.2}$$

i.e. it satisfies the profile equation

$$k^2\phi'' + kc\phi' + f(\phi) = 0. \tag{4.1.3}$$

Given such a solution, note that a function of the form $u(x,t) = \phi(x) + v(x,t)$ is a solution of (4.1.2) provided it satisfies a system of the form

$$kv_t = k\mathcal{L}[\phi]v + \mathcal{N}(v), \tag{4.1.4}$$

where here $\mathcal{N}(v)$ is at least quadratic in $v$ and $\mathcal{L}[\phi]$ is the linear differential operator

$$k\mathcal{L}[\phi] := k^2\partial_x^2 + kc\partial_x + Df(\phi).$$

Naturally, the domain of the operator $\mathcal{L}[\phi]$ is determined by the chosen class of perturbations $v$ of the underlying standing wave $\phi$ and, as mentioned above, several choices are available in the literature. As we are interested in subharmonic perturbations, i.e., perturbations with period $N \in \mathbb{N}$, we consider $\mathcal{L}[\phi]$ as a closed, densely defined linear operator acting on $L^2_{\text{per}}(0, N)$ with 1-periodic coefficients.

The stability analysis of periodic waves to such subharmonic perturbations naturally relies on a detailed understanding of the spectrum of $\mathcal{L}[\phi]$ acting on $L^2_{\text{per}}(0, N)$. To describe the $N$-periodic spectrum of $\mathcal{L}[\phi]$, we begin by recalling the notion of spectral stability given in Definition 1.3.1 that has been used throughout this dissertation.

**Definition 4.1.1.** A 1-periodic stationary solution $\phi \in H^1_{\text{loc}}(\mathbb{R})$ of (4.1.2) is said to be *diffusively spectrally stable* provided the following conditions hold:

(i) The spectrum of the linear operator $\mathcal{L}[\phi]$ acting on $L^2(\mathbb{R})$ satisfies

$$\sigma_{L^2(\mathbb{R})}\left(\mathcal{L}[\phi]\right) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\} \cup \{0\};$$

(ii) There exists a $\theta > 0$ such that for any $\xi \in [-\pi, \pi)$ the real part of the spectrum of the Bloch operator $\mathcal{L}_\xi[\phi] := e^{-i\xi x}\mathcal{L}[\phi]e^{i\xi x}$ acting on $L^2_{\text{per}}(0, 1)$ satisfies

$$\Re\left(\sigma_{L^2_{\text{per}}(0,1)}\left(\mathcal{L}_\xi[\phi]\right)\right) \leq -\theta\xi^2;$$

(iii) $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_0[\phi]$ with associated eigenfunction $\phi'$.

As mentioned in Section 1.3.2, the above notion of spectral stability has been taken as the standard spectral assumption in nonlinear stability results for periodic traveling/standing waves in reaction diffusion systems since the pioneering work of Schnieder [58, 59]. Specifically, the above notion of spectral stability is sufficiently strong to allow one to immediately conclude important details regarding the nonlinear dynamics of $\phi$ under localized, or general bounded, perturbations, including long-time asymptotics of the associated modulation functions. For more information, see [13, 29, 30, 56, 57] and references therein.

**Remark 4.1.2.** *Note the assumption on simplicity of the eigenvalue* $\lambda = 0$ *is natural since such periodic standing waves typically appear as one-parameter families parametrized only by translational invariance. Indeed, solutions of* (4.1.3) *are readily seen to rely (up to translation invariance) on the* $n + 2$ *parameters* $(\phi(0), k, c)$*, while periodicity requires the enforcement of* $n$ *constraints, leaving in general a two-parameter family of* 1*-periodic solutions*

$$u(x - ct - x_0; c, x_0)$$

*which satisfy* (4.1.3) *with* $k = k(c)$*. Due to the secular dependence of the frequency* $k$ *on the wave speed* $c$*, variations in* $c$ *do not preserve periodicity and hence, generically, it follows one should expect the kernel of* $\mathcal{L}[\phi]$ *to be one-dimensional, which leads to (iii) in Definition* (1.3.1) *above.*

Given a diffusively spectrally stable 1-periodic traveling wave solution $\phi$ of (4.1.1), one can now easily characterize the spectrum of $\mathcal{L}[\phi]$ acting on $L^2_{\text{per}}(0, N)$. Indeed, as described in Section 1.3.4, the spectrum of $\mathcal{L}[\phi]$ acting on $L^2_{\text{per}}(0, N)$ is equal to the union of the necessarily discrete[1] spectrum of the corresponding Bloch operators $\mathcal{L}_\xi[\phi]$ acting on $L^2_{\text{per}}(0, 1)$ for $\xi \in \Omega_N$, which is defined in (1.3.10) as the discrete (finite) subset of $\xi \in [-\pi, \pi)$ such that $e^{i\xi N} = 1$. As discussed in Remark 1.3.3, it follows that diffusively spectrally stable periodic traveling waves of (4.1.1) are necessarily spectrally stable to all subharmonic perturbations. In particular, for each $N \in \mathbb{N}$ the non-zero $N$-periodic eigenvalues of $\mathcal{L}[\phi]$ satisfy the spectral gap condition

$$\Re\left(\sigma_{L^2_{\text{per}}(0,N)}\left(\mathcal{L}[\phi]\right) \setminus \{0\}\right) \le -\delta_N$$

[1]Note since the domains of the operators $\mathcal{L}_\xi[\phi]$ are compactly contained in $L^2_{\text{per}}(0, 1)$, it follows that their $L^2_{\text{per}}(0, 1)$-spectrum is comprised entirely of isolated eigenvalues with finite multiplicities.

for some constant $\delta_N > 0$. From here, using that $\mathcal{L}[\phi]$ is sectorial, it is easy to show that for each $\delta \in (0, \delta_N)$ there exists a constant $C_\delta > 0$ such that

$$\left\| e^{\mathcal{L}[\phi]t} \left(1 - \mathcal{P}_1\right) f \right\|_{L^2_{\mathrm{per}}(0,N)} \leq C_\delta e^{-\delta t} \|f\|_{L^2_{\mathrm{per}}(0,N)}. \tag{4.1.5}$$

for all $f \in L^2_{\mathrm{per}}(0, N)$, where here $\mathcal{P}_1$ denotes the projection of $L^2_{\mathrm{per}}(0, N)$ onto the $N$-periodic kernel of $\mathcal{L}[\phi]$ spanned by $\phi'$. Equipped with this linear estimate, one can now establish the following nonlinear stability result.

**Proposition 4.1.3.** *Let $\phi \in H^1_{\mathrm{loc}}$ be a 1-periodic stationary solution of* (4.1.2) *and fix $N \in \mathbb{N}$. Assume that $\phi$ is diffusively spectrally stable, in the sense of Definition 1.3.1 above and, for each $N \in \mathbb{N}$, take $\delta_N > 0$ such that*

$$\max \Re \left(\sigma_{L^2_{\mathrm{per}}(0,N)} \left(\mathcal{L}[\phi]\right) \setminus \{0\}\right) = -\delta_N \tag{4.1.6}$$

*holds. Then for each $N \in \mathbb{N}$, $\phi$ is asymptotically stable to subharmonic $N$-periodic perturbations. More precisely, for every $\delta \in (0, \delta_N)$ there exists an $\varepsilon = \varepsilon_\delta > 0$ and a constant $C = C_\delta > 0$ such that whenever $u_0 \in H^1_{\mathrm{per}}(0, N)$ and $\|u_0 - \phi\|_{H^1(0,N)} < \varepsilon$, then the solution $u$ of* (4.1.2) *with initial data $u(0) = u_0$ exists globally in time and satisfies*

$$\|u(\cdot, t) - \phi(\cdot + \sigma_\infty)\|_{H^1(0,N)} \leq Ce^{-\delta t} \|u_0 - \phi\|_{H^1(0,N)}$$

*for all $t > 0$, where here $\sigma_\infty = \sigma_\infty(N)$ is some constant.*

The proof of Proposition 4.1.3 is by now standard, and can be completed by following appropriate texts: see, for example, [40, Chapter 4]. The main idea is that the linear estimate (4.1.5) suggests that if $u(x, t)$ is a solution of (4.1.2) which is initially close to $\phi$ in $L^2_{\mathrm{per}}(0, N)$, then there exists a (small) time-dependent modulation function $\sigma(t)$ such that $u(x, t)$ essentially behaves for large time as

$$u(x, t) \approx \phi(x) + \sigma(t)\phi'(x) \approx \phi(x + \sigma(t)),$$

corresponding to standard asymptotic (orbital) stability of $\phi$. With this insight gained from (4.1.5), a straightforward nonlinear iteration scheme completes the proof of Proposition 4.1.3.

While Proposition 4.1.3 establishes nonlinear stability of $\phi$ in $L^2_{\mathrm{per}}(0,N)$ for each fixed $N \in \mathbb{N}$, it lacks uniformity in $N$ in two important (and related) aspects. Indeed, note that the exponential rate of decay $\delta$ and the allowable size of initial perturbations $\varepsilon = \varepsilon_\delta$ are both controlled completely in terms of the size of the spectral gap $\delta_N > 0$. Since $\delta_N \to 0$ as $N \to \infty$, it follows that both $\delta$ and $\varepsilon$ chosen in Proposition 4.1.3 necessarily tend to zero[2] as $N \to \infty$. With this observation in mind, it is natural to ask if one can obtain a stability result to $N$-periodic perturbations which is uniform in $N$. In such a result, one should naturally require that both the rate of decay and and the size of initial perturbations be independent of $N$, thus depending only on the background wave $\phi$. This is precisely achieved in our main result.

**Theorem 4.1.4** (Uniform Subharmonic Asymptotic Stability)**.** *Fix*[3] $K \geq 3$*. Suppose* $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ *is a* 1*-periodic stationary solution of* (4.1.2) *that is diffusively spectrally stable, in the sense of Definition 1.3.1. There exists an* $\varepsilon > 0$ *and a constant* $C > 0$ *such that, for each* $N \in \mathbb{N}$*, whenever* $u_0 \in L^1_{\mathrm{per}}(0,N) \cap H^K_{\mathrm{per}}(0,N)$ *and*

$$E_0 := \|u_0 - \phi\|_{L^1_{\mathrm{per}}(0,N) \cap H^K_{\mathrm{per}}(0,N)} < \varepsilon,$$

*there exists a function* $\widetilde{\psi}(x,t)$ *satisfying* $\widetilde{\psi}(\cdot,0) \equiv 0$ *such that the solution of* (4.1.2) *with initial data* $u(0) = u_0$ *exists globally in time and satisfies*

$$\left\| u\left(\cdot - \widetilde{\psi}(\cdot,t), t\right) - \phi \right\|_{H^K_{\mathrm{per}}(0,N)}, \quad \left\|\nabla_{x,t}\widetilde{\psi}(\cdot,t)\right\|_{H^K_{\mathrm{per}}(0,N)} \leq C E_0 (1+t)^{-3/4} \tag{4.1.7}$$

*for all* $t \geq 0$*. Further, there exists constants* $\gamma_\infty \in \mathbb{R}$ *and* $C > 0$ *such that for each* $N \in \mathbb{N}$ *we have*

$$\left\| \widetilde{\psi}(\cdot,t) - \frac{1}{N}\gamma_\infty \right\|_{H^K_{\mathrm{per}}(0,N)} \leq C E_0 (1+t)^{-1/4} \tag{4.1.8}$$

[2]Additionally, this degeneracy can be seen in the linear estimate (4.1.5) since both $\delta \to 0^+$ and $C_\delta \to \infty$ as $N \to \infty$.
[3]Here and throughout, $K$ encodes the regularity of the nonlinearity $f$ in (4.1.1).

*for all $t \geq 0$.*

**Remark 4.1.5.** *Using the methods in [29, 37], the results in Theorem 4.1.4 can easily be extended to establish uniform (in $N$) decay rates of perturbations in $L^p_{\text{per}}(0, N)$ for any $2 \leq p \leq \infty$ provided the initial perturbations are again sufficiently small in $L^1_{\text{per}}(0, N) \cap H^K_{\text{per}}(0, N)$. For simplicity, however, and to establish proof of concept, in this work we concentrate on the $L^2$-based theory only.*

The key idea to the proof of Theorem 4.1.4 is to use the stability theory of periodic waves of reaction diffusion equations to *localized perturbations*, specifically those techniques developed in [29, 37], as a guide for how to uniformly control the dynamics of subharmonic perturbations for large $N$. Indeed, observe the decay rates guaranteed in Theorem 4.1.4 are precisely those predicted by considering the dynamics of such periodic wave trains to localized perturbations: see [31, 29, 30, 37], for example. Formally, this should not be too surprising since, up to appropriate translations, a sequence of $N$-periodic functions may converge (locally) as $N \to \infty$ to functions in $L^2(\mathbb{R})$.

We make the above intuition precise by first following the methodology recently developed in [21] (see also Chapter 3) in order to provide a delicate decomposition of the semigroup $e^{\mathcal{L}[\phi]t}$ acting on the the space $L^2_{\text{per}}(0, N)$ with $N \in \mathbb{N}$. This decomposition is accomplished by adapting the linear theory for localized perturbations developed in [29, 37] to the subharmonic context in order to uniformly handle the accumulation of Bloch eigenvalues near the origin as $N \to \infty$. Furthermore, our linear decomposition, which will be reviewed in Section 4.3 below, not only recovers the exponential decay rates exhibited in Proposition 4.1.3, but they also provide the uniform (in $N$) rates of decay in Theorem 4.1.4. As we will see, this linear analysis predicts that if $u(x, t)$ is a solution of (4.1.2) which is initially close to $\phi$ in $L^2_{\text{per}}(0, N)$ then there exists a (small) space-time dependent, $N$-periodic (in $x$) modulation function $\widetilde{\psi}(x, t)$ such that $u(x, t)$ essentially behaves for large time like

$$u(x, t) \approx \phi(x) + \widetilde{\psi}(x, t)\phi'(x) \approx \phi\left(x + \widetilde{\psi}(x, t)\right)$$

giving a refined insight into the long-time local dynamics near $\phi$ beyond the more standard asymptotic stability (with asymptotic phase) as in Proposition 4.1.3. Motivated by this initial linear analysis, we then build a nonlinear iteration scheme for subharmonic perturbations which incorporates phase modulation functions which depend on *both space and time* in order to complete the proof of Theorem 4.1.4. The requirement that the modulation functions are spatially dependent is necessary for our method, and is fundamentally different than the methodology used in the proof of Proposition 4.1.3. In particular, to the authors' knowledge, this work is the first to consider spatially dependent modulation functions in the context of periodic perturbations. Furthermore, Theorem 4.1.4 is the first result to obtain stability results for periodic waves to subharmonic perturbations that are uniform in the period of the perturbation.

**Remark 4.1.6.** *As indicated above, the strategy for proving our subharmonic results follows the stability analyses [29, 37] for localized perturbations of periodic wave trains in reaction diffusion systems. In the localized case, the origin is always a part of the essential spectrum of the linearized operator, leading one to introduce space-time dependent modulation functions. In the subharmonic case, however, the origin is an isolated simple eigenvalue for each fixed $N \in \mathbb{N}$, and using time-dependent modulations only leads to results such as Proposition 4.1.3. In order to achieve the proof of Theorem 4.1.4, we will rely on a combination of these approaches, using an $N$-dependent time-modulation function to account for the isolated eigenvalue at the origin, while simultaneously using a space-time modulation to account for the accumulation of spectrum near the origin as $N \to \infty$.*

Next, we point out an important corollary of Theorem 4.1.4. Particularly, since the decay rates in Theorem 4.1.4 are sufficiently fast we can obtain the following result accounting for only time-dependent modulations yet offering slower uniform decay rates. Note that while the result uses only time-dependent modulations, the proof requires the use of space-time dependent modulation functions.

**Corollary 4.1.7.** *Under the hypotheses of Theorem 4.1.4, there exists an $\varepsilon > 0$ and a constant $C > 0$ such that, for each $N \in \mathbb{N}$, whenever $u_0 \in L^1_{\text{per}}(0, N) \cap H^K_{\text{per}}(0, N)$ and $E_0 < \varepsilon$, there exists* *a function $\gamma(t)$ satisfying $\gamma(0) = 0$ such that the solution $u$ of* (4.1.2) *with initial data $u(0) = u_0$ exists globally in time and satisfies*

$$\left\| u\left(\cdot - \frac{1}{N}\gamma(t), t\right) - \phi \right\|_{H^K_{\text{per}}(0,N)} \leq CE_0(1+t)^{-1/4}. \tag{4.1.9}$$

*for all $t > 0$. Further, the time-dependent modulation function $\gamma(t)$ satisfies*

$$|\gamma_t(t)| \leq CE_0(1+t)^{-3/2}$$

*and hence, in particular, there exists a $\gamma_\infty \in \mathbb{R}$*

$$|\gamma(t) - \gamma_\infty| \leq CE_0(1+t)^{-1/2}$$

*for all $t > 0$. In particular,*

$$\left\| u(\cdot, t) - \phi\left(\cdot + \frac{1}{N}\gamma_\infty\right) \right\|_{H^K_{\text{per}}(0,N)} \leq CE_0(1+t)^{-1/4}.$$

*for all $t > 0$.*

**Remark 4.1.8.** *Comparing Corollary 4.1.7 with Proposition 4.1.3, we see that one necessarily has the relationship $\sigma_\infty(N) = \frac{1}{N}\gamma_\infty$, establishing a direct correspondence between the (N-dependent) asymptotic phase shifts. Further, we note the $\gamma_\infty$ is the same in both Theorem 4.1.4 and Corollary 4.1.7.*

Our last result combines the results of Corollary 4.1.7 with Proposition 4.1.3 in order to obtain a nonlinear stability result allowing a uniform (in $N$) size of initial perturbations with (eventual) exponential rates of decay.

**Corollary 4.1.9.** *Under the hypotheses of Theorem 4.1.4, there exists an $\varepsilon > 0$ and a constant $C > 0$ such that, for each $N \in \mathbb{N}$ and $\delta \in (0, \delta_N)$, with $\delta_N$ as in* (4.1.6)*, whenever $u_0 \in$*

$L^1_{\mathrm{per}}(0,N) \cap H^K_{\mathrm{per}}(0,N)$ *with* $E_0 < \varepsilon$ *there exists a* $T_\delta > 0$ *and a constant* $M_\delta > 0$ *such that*

$$\left\| u(\cdot,t) - \phi\left(\cdot + \frac{1}{N}\gamma_\infty\right) \right\|_{H^1_{\mathrm{per}}(0,N)} \le \begin{cases} CE_0(1+t)^{-1/4}, & \text{for } 0 < t \le T_\delta \\ M_\delta E_0 e^{-\delta t}, & \text{for } t > T_\delta. \end{cases}$$

The above corollary has a few important features to highlight. First, we emphasize that $\varepsilon$, the size of the initial perturbation above, is independent of both $N$ and the choice $\delta \in (0, \delta_N)$. In particular, this establishes a uniform size on the domain of attraction for perturbations to (eventually) exhibit exponential decay. This is in stark contrast to Proposition 4.1.3 which requires $\varepsilon_\delta \to 0$ as $\delta \to 0$. Secondly, we note that the length of time one must wait to observe exponential decay, quantified by $T_\delta$ above, necessarily satisfies $T_\delta \to \infty$ as $\delta \to 0$; hence, it is not uniform in $N$. Nevertheless, Corollary 4.1.9 upgrades the long-time behavior of Proposition 4.1.3 allowing for a uniform size of initial perturbations. Interestingly, Corollary 4.1.9 can be easily seen, at least at the linear level, directly from our forthcoming decomposition of the semigroup $e^{\mathcal{L}[\phi]t}$: see Remark 4.3.3 in Section 4.3 below.

The online of the chapter is as follows. In Section 4.2, we collect several properties of the Bloch operators and their associated semigroups. In particular, we establish basic decay properties of the Bloch semigroups arising as a result of the diffusive spectral stability assumption. In Section 4.3, we review and expand upon our key linear estimates from Section 3.4 by providing a delicate decomposition of the semigroup $e^{\mathcal{L}[\phi]t}$ acting on $L^2_{\mathrm{per}}(0,N)$, which allows us to identify polynomial decay rates on the linear evolution which are *uniform in* $N$: see Proposition 4.3.1. These linear estimates form the backbone for our nonlinear analysis, which is detailed in Section 4.4. In Section 4.4.1, we use intuition gained from the linear estimates of Section 4.3 to introduce an appropriate nonlinear decomposition of a small $L^2_{\mathrm{per}}(0,N)$ neighborhood of the underlying diffusively stable 1-periodic wave $\phi$, and we develop appropriate perturbation equations satisfied by the corresponding perturbation and modulation functions. In Section 4.4.2, we apply a nonlinear iteration scheme to the system of perturbation equations obtained in Section 4.4.1 and present the proofs of Theorem

4.1.4 and its corollaries stated above. Finally, a proof of some technical results from Section 4.3 are provided in Appendix 4.A.

## 4.2 Spectral Preparation & Properties of Bloch Semigroups

In this section, we prove several preliminary results. In particular, we establish some elementary semigroup estimates for the associated Bloch operators. Throughout the remainder of the chapter, for notational convenience, we set for each $N \in \mathbb{N}$ and $p \geq 1$

$$L^p_N := L^p_{\text{per}}(0, N).$$

With the characterization of the $L^2_N$-spectrum of the linearized operator $\mathcal{L}[\phi]$ about a 1-periodic stationary solution $\phi$ of (4.1.2) provided in Section 1.3.4, we can now provide some immediate consequences of the diffusive spectral stability assumption in Definition 1.3.1. Specifically, in our present subharmonic context we recall from Remark 1.3.3 that if $\phi$ is such a diffusively spectrally stable standing solution of (4.1.2), then for each $N \in \mathbb{N}$ there exists a $\delta_N > 0$ such that

$$\Re\left(\sigma_{L^2_N}\left(\mathcal{L}[\phi]\right) \setminus \{0\}\right) \leq -\delta_N,$$

i.e., the non-zero $N$-periodic eigenvalues of $\mathcal{L}[\phi]$ are uniformly bounded away from the imaginary axis. In particular, by standard spectral perturbation theory, we immediately have that the following spectral properties hold.

**Lemma 4.2.1** (Spectral Preparation)**.** *Suppose that $\phi$ is a* 1*-periodic stationary solution of* (4.1.2) *which is diffusively spectrally stable. Then the following properties hold.*

*(i) For any fixed $\xi_0 \in (0, \pi)$, there exists a constant $\delta_0 > 0$ such that*

$$\Re\left(\sigma\left(\mathcal{L}_\xi[\phi]\right)\right) < -\delta_0$$

*for all $\xi \in [-\pi, \pi)$ with $|\xi| > \xi_0$.*

*(ii) There exist positive constants $\xi_1$ and $\delta_1$ such that for any $|\xi| < \xi_1$, the spectrum of $\mathcal{L}_\xi[\phi]$ decomposes into two disjoint subsets*

$$\sigma\left(\mathcal{L}_\xi[\phi]\right) = \sigma_-\left(\mathcal{L}_\xi[\phi]\right) \bigcup \sigma_0\left(\mathcal{L}_\xi[\phi]\right)$$

*with the following properties:*

*(a) $\Re\, \sigma_-\left(\mathcal{L}_\xi[\phi]\right) < -\delta_1$ and $\Re\, \sigma_0\left(\mathcal{L}_\xi[\phi]\right) > -\delta_1$;*

*(b) the set $\sigma_0\left(\mathcal{L}_\xi[\phi]\right)$ consists of a single eigenvalue $\lambda_c(\xi)$ which is analytic in $\xi$ and expands as*

$$\lambda_c(\xi) = ia\xi - d\xi^2 + O(\xi^3)$$

*for $|\xi| \ll 1$ and some constants $a \in \mathbb{R}$ and $d > 0$;*

*(c) the eigenfunction associated to $\lambda_c(\xi)$ is analytic near $\xi = 0$ and expands as*

$$\Phi_\xi(x) = \phi'(x) + O(\xi)$$

*for $|\xi| \ll 1$.*

The proof of (i) follows immediately from the properties (i) and (ii) in Definition 1.3.1, while the second part follows since $\lambda = 0$ is a simple eigenvalue of the co-periodic operator $\mathcal{L}_0[\phi]$ and that the coefficients of $\mathcal{L}_\xi[\phi]$ clearly vary analytically on $\xi$.

With the above spectral preparation result in hand, we now record some key induced features of the associated semigroups. These estimates are immediate consequences of Lemma 4.2.1 and the fact that the Bloch operators are clearly sectorial when acting on $L^2_{\text{per}}(0,1)$.

**Proposition 4.2.2.** *Suppose that $\phi$ is a 1-periodic stationary solution of (4.1.2) which is diffusively spectrally stable. Then the following properties hold.*

*(i) For any fixed $\xi_0 \in (0, \pi)$, there exist positive constants $C_0$ and $d_0$ such that*

$$\left\| e^{\mathcal{L}_\xi[\phi]t} f \right\|_{B(L^2_{\mathrm{per}}(0,1))} \leq C_0 e^{-d_0 t}$$

*valid for all $t \geq 0$ and all $\xi \in [-\pi, \pi)$ with $|\xi| > \xi_0$.*

*(ii) With $\xi_1$ chosen as in Lemma 4.2.1, there exist positive constants $C_1$ and $d_1$ such that for any $|\xi| < \xi_1$, if $\Pi(\xi)$ denotes the (rank-one) spectral projection onto the eigenspace associated to $\lambda_c(\xi)$ given by Lemma 4.2.1(ii), then*

$$\left\| e^{\mathcal{L}_\xi[\phi]t} (1 - \Pi(\xi)) \right\|_{B(L^2_{\mathrm{per}}(0,1))} \leq C_1 e^{-d_1 t}$$

*for all $t \geq 0$.*

Coupled with an appropriate decomposition of $e^{\mathcal{L}[\phi]t}$, the above linear estimates form the core of our forthcoming linear analysis (which, in turn, forms the backbone of our nonlinear iteration scheme).

## 4.3 Uniform Subharmonic Linear Estimates

We begin our analysis by obtaining decay rates on the semigroup $e^{\mathcal{L}[\phi]t}$ acting on classes of subharmonic perturbations in $L^2_N$ which are uniform in $N$. This analysis is based on a delicate decomposition of the semigroup. In particular, we use (1.3.16) to study the action of $e^{\mathcal{L}[\phi]t}$ on $L^2_N$ in terms of associated Bloch operators, which is accomplished by separating the semigroup into appropriate critical frequency and non-critical frequency components. Note that, due to Lemma 4.2.1 we expect the "critical frequency" component to be dominated by the translational mode $\phi'$. This decomposition was recently carried out in detail (in a related context) in [21] (see also Chapter 3),and for completeness we review it here. Note the decomposition is heavily motivated by the corresponding decomposition used in the case of localized perturbations: see [31, 29].

To begin, let $\xi_1 \in (0, \pi)$ be defined as in Lemma 4.2.1 and let $\rho$ be a smooth cutoff function satisfying $\rho(\xi) = 1$ for $|\xi| < \frac{\xi_1}{2}$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$. For a given $v \in L^2_N$, we use (1.3.16) to decompose $e^{\mathcal{L}[\phi]t}$ into low-frequency and high-frequency components as

$$\begin{aligned} e^{\mathcal{L}[\phi]t}v(x) &= \frac{1}{N}\sum_{\xi\in\Omega_N} \rho(\xi)e^{i\xi x}e^{\mathcal{L}_\xi[\phi]t}\mathcal{B}_1(v)(\xi,x) \\ &\qquad\qquad + \frac{1}{N}\sum_{\xi\in\Omega_N} (1-\rho(\xi))\, e^{i\xi x}e^{\mathcal{L}_\xi[\phi]t}\mathcal{B}_1(v)(\xi,x) \\ &= S_{lf,N}(t)v(x) + S_{hf,N}v(x). \end{aligned} \tag{4.3.1}$$

Using Proposition 4.2.2 and the subharmonic Parseval identity 1.3.15, it follows that there exist constants $C, \eta > 0$, both independent of $N$, such that

$$\begin{aligned} \left\|S_{hf,N}(t)v\right\|^2_{L^2_N} &= \frac{1}{N}\sum_{\xi\in\Omega_N} \left\|(1-\rho(\xi))e^{\mathcal{L}_\xi[\phi]t}\mathcal{B}_1(v)(\xi,\cdot)\right\|^2_{L^2(0,1)} \\ &\le \frac{1}{N}\sum_{\xi\in\Omega_N} (1-\rho(\xi))^2\left\|e^{\mathcal{L}_\xi[\phi]t}\right\|^2_{B(L^2(0,1))}\|\mathcal{B}_1(v)(\xi,\cdot)\|^2_{L^2(0,1)} \\ &\le Ce^{-2\eta t}\left(\frac{1}{N}\sum_{\xi\in\Omega_N}\|\mathcal{B}_1(v)(\xi,\cdot)\|^2_{L^2(0,1)}\right), \end{aligned}$$

which, again using Parseval's identity (1.3.15), yields the exponential decay estimate

$$\left\|S_{hf,N}(t)v\right\|_{L^2_N} \le Ce^{-\eta t}\|v\|_{L^2_N}. \tag{4.3.2}$$

For the low-frequency component, for each $|\xi| < \xi_1$ define the rank-one spectral projection onto the critical mode of $\mathcal{L}_\xi[\phi]$ by

$$\begin{cases} \Pi(\xi) : L^2_{\text{per}}(0,1) \to \ker\left(\mathcal{L}_\xi[\phi] - \lambda_c(\xi)I\right) \\ \Pi(\xi)g(x) = \left\langle\widetilde{\Phi}_\xi, g\right\rangle_{L^2(0,1)}\Phi_\xi(x) \end{cases}$$

where here $\widetilde{\Phi}_\xi$ denotes the element of the kernel of the adjoint $\mathcal{L}_\xi[\phi]^\dagger - \overline{\lambda_c(\xi)}I$ satisfying the normalization condition $\left\langle\widetilde{\Phi}_\xi, \Phi_\xi\right\rangle_{L^2(0,1)} = 1$. The low-frequency operator $S_{lf,N}$ can thus be further decomposed into the contribution from the critical mode and the contribution from low-frequency spectrum bounded away from $\lambda = 0$ via

$$
\begin{aligned}
S_{lf,N}(t)v(x) = \frac{1}{N}\sum_{\xi\in\Omega_N} \rho(\xi)e^{i\xi x}e^{\mathcal{L}_\xi[\phi]t}\Pi(\xi)\mathcal{B}_1(v)(\xi,x) \\
+ \frac{1}{N}\sum_{\xi\in\Omega_N} \rho(\xi)e^{i\xi x}e^{\mathcal{L}_\xi[\phi]t}\left(1-\Pi(\xi)\right)\mathcal{B}_1(v)(\xi,x) \\
=: S_{c,N}v(x) + \widetilde{S}_{lf,N}(t)v(x).
\end{aligned}
\tag{4.3.3}
$$

As with the exponential estimate (4.3.2), Proposition 4.2.2 implies, by possibly choosing $\eta > 0$ smaller, that there exists a constant $C > 0$ independent of $N$ such that

$$
\left\|\widetilde{S}_{lf,N}(t)v\right\|_{L^2_N} \leq Ce^{-\eta t}\|v\|_{L^2_N}. \tag{4.3.4}
$$

For the critical component $S_{c,N}$, note by Lemma 4.2.1(ii) that we can write

$$
\begin{aligned}
S_{c,N}(t)v(x) &= \frac{1}{N}e^{\mathcal{L}_0[\phi]t}\Pi(0)\mathcal{B}_1(v)(0,x) \\
&\quad + \frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \rho(\xi)e^{i\xi x}e^{\mathcal{L}_\xi[\phi]t}\Pi(\xi)\mathcal{B}_1(v)(\xi,x) \\
&= \frac{1}{N}\phi'(x)\left\langle\widetilde{\Phi}_0, \mathcal{B}_1(v)(0,\cdot)\right\rangle_{L^2(0,1)} \\
&\quad + \frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \rho(\xi)e^{i\xi x}e^{\lambda_c(\xi)t}\Phi_\xi(x)\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)}.
\end{aligned}
$$

and hence, recalling Lemma 1.3.5 and expanding $\Phi_\xi$,

$$
\begin{aligned}
S_{c,N}(t)v(x) &= \frac{1}{N}\phi'(x)\left\langle\widetilde{\Phi}_0, v\right\rangle_{L^2_N} + \phi'(x)\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \rho(\xi)e^{i\xi x}e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)} \\
&\quad + \frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \rho(\xi)e^{i\xi x}(i\xi)e^{\lambda_c(\xi)t}\left(\frac{\widetilde{\Phi}_\xi(x)-\phi'(x)}{i\xi}\right)\left\langle\widetilde{\Phi}_\xi, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)} \\
&=: \frac{1}{N}\phi'(x)\left\langle\widetilde{\Phi}_0, v\right\rangle_{L^2_N} + \phi'(x)s_{p,N}(t)v(x) + \widetilde{S}_{c,N}(t)v(x).
\end{aligned}
$$

Taken together, it follows that the linear solution operator $e^{\mathcal{L}[\phi]t}$ can be decomposed as

$$e^{\mathcal{L}[\phi]t}v(x) = \frac{1}{N}\phi'(x)\left\langle \widetilde{\Phi}_0, v\right\rangle_{L^2_N} + \phi'(x)s_{p,N}(t)v(x) + \widetilde{S}_N(t)v(x) \tag{4.3.5}$$

where

$$s_{p,N}(t)v(x) = \frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \rho(\xi)e^{i\xi x}e^{\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)}$$

and

$$\widetilde{S}_N(t)v(x) = S_{hf,N}(t)v(x) + \widetilde{S}_{lf,N}(t)v(x) + \widetilde{S}_{c,N}(t)v(x).$$

Equipped with the above, we can establish our main set of linear estimates.

**Proposition 4.3.1** (Linear Estimates)**.** *Suppose that $\phi$ is a 1-periodic stationary solution of* (4.1.2) *which is diffusively spectrally stable. Given any $M\in\mathbb{N}$, there exists a constant $C>0$ such that for all $t\geq 0$, $N\in\mathbb{N}$ and all $0\leq l,m\leq M$ we have*

$$\left\|\partial_x^l\partial_t^m s_{p,N}(t)v\right\|_{L^2_N} \leq C(1+t)^{-1/4-(l+m)/2}\|v\|_{L^1_N}$$

*Furthermore, there exists constants $C,\eta>0$ such that for all $t\geq 0$ and $N\in\mathbb{N}$ we have*

$$\left\|\widetilde{S}_N(t)v\right\|_{L^2_N} \leq C\left((1+t)^{-3/4}\|v\|_{L^1_N} + e^{-\eta t}\|v\|_{L^2_N}\right).$$

**Remark 4.3.2.** *While the bounds above on the derivatives of $s_{p,N}(t)$ are largely unmotivated by our linear analysis, they will be essential in our forthcoming nonlinear theory.*

*Proof.* First observe that, by definition of $\mathcal{B}_1$, we have

$$\begin{aligned}\left\langle \widetilde{\Phi}_\xi, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)} &= \int_0^1 \overline{\widetilde{\Phi}_\xi(x)}\sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x}\widehat{v}(\xi+2\pi\ell)dx \\ &= \sum_{\ell\in\mathbb{Z}}\widehat{v}(\xi+2\pi\ell)\int_0^1 \overline{\widetilde{\Phi}_\xi(x)}e^{2\pi i\ell x}dx\end{aligned}$$

$$= \sum_{\ell\in\mathbb{Z}} \widehat{v}(\xi+2\pi\ell)\overline{\widehat{\widetilde{\Phi}_{\xi}}(2\pi\ell)}$$

and hence, using the fact that (1.3.12) implies $\|\widehat{v}\|_{L^\infty(\mathbb{R})} \leq \|v\|_{L^1_N}$ along with Cauchy-Schwartz, it follows that

$$\begin{aligned}\rho(\xi)\left|\left\langle\widetilde{\Phi}_{\xi}, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)}\right|^2 &\leq \rho(\xi)\|v\|_{L^1_N}^2\left(\sum_{\ell\in\mathbb{Z}}(1+|\ell|^2)^{1/2}\left|\overline{\widehat{\widetilde{\Phi}_{\xi}}(2\pi\ell)}\right|(1+|\ell|^2)^{-1/2}\right)^2\\ &\leq C\|v\|_{L^1_N}^2\sup_{\xi\in[-\pi,\pi)}\left(\rho(\xi)\left\|\widetilde{\Phi}_{\xi}\right\|_{H^1_{\mathrm{per}}(0,1)}^2\right).\end{aligned}$$

valid for all $\xi\in\Omega_N$. Using Lemma 4.2.1, it follows by Parseval's identity (1.3.15) that there exists constants $C, d > 0$, independent of $N$, such that

$$\begin{aligned}\left\|\partial_x^l\partial_t^m s_{p,N}(t)v\right\|_{L^2_N}^2 &= \frac{1}{N}\sum_{\xi\in\Omega_N}\left\|\rho(\xi)(i\xi)^l\,(\lambda_c(\xi))^m\,e^{\lambda_c(\xi)t}\left\langle\widetilde{\Phi}_{\xi}, \mathcal{B}_1(v)(\xi,\cdot)\right\rangle_{L^2(0,1)}\right\|_{L^2(0,1)}^2\\ &\leq C\|v\|_{L^1_N}^2\left(\frac{1}{N}\sum_{\xi\in\Omega_N}|\xi|^{2(l+m)}e^{-2d\xi^2 t}\right).\end{aligned} \tag{4.3.6}$$

By similar considerations, we find that

$$\left\|\widetilde{S}_N(t)v\right\|_{L^2_N}^2 \leq Ce^{-2\eta t}\|v\|_{L^2_N}^2 + C\|v\|_{L^1_N}^2\left(\frac{1}{N}\sum_{\xi\in\Omega_N}|\xi|^2 e^{-2d\xi^2 t}\right). \tag{4.3.7}$$

It remains to provide uniform in $N$ decay rates on the finite sums

$$\frac{1}{N}\sum_{\xi\in\Omega_N}|\xi|^{2(l+m)}e^{-2d\xi^2 t}\ \text{ and }\ \frac{1}{N}\sum_{\xi\in\Omega_N}|\xi|^2 e^{-2d\xi^2 t}. \tag{4.3.8}$$

To gain some intuition on how to uniformly bound these sums, notice that they can be interpreted as Riemann sum approximations (up to a harmless rescaling) of the integrals

$$\int_{-\pi}^{\pi}\xi^{2(\ell+m)}e^{-2d\xi^2 t}d\xi, \qquad \int_{-\pi}^{\pi}\xi^2 e^{-2d\xi^2 t}d\xi, \tag{4.3.9}$$

which, through an elementary scaling argument, exhibit $(1+t)^{-1/2-(\ell+m)}$ and $(1+t)^{-3/2}$ decay for large time, respectively. The proof that the Riemann sums are uniformly controlled by these decay rates is provided in Lemma 4.A.1 in Appendix 4.A, which completes the proof. □

**Remark 4.3.3.** *The result of Corollary 4.1.9 can be seen from the above analysis, at least at the linear level. Indeed, following the methods in [21, Section 5] (see also Chapter 3) one sees that, for large $N$, the sums in* (4.3.8) *are good approximations of the respective integrals in* (4.3.9) *for times up to $t = O(N^2)$, corresponding to an observed polynomial decay of perturbations on such a timescale. For larger times, however, the exponential nature of the summands dominate and the sums decay monotonically to zero at exponential rates, corresponding to an exponential decay of perturbations on these longer timescales.*

Before continuing to our nonlinear analysis, we pause to interpret the above results. Suppose that $\phi$ is a 1-periodic diffusively spectrally stable stationary solution of (4.1.2), and let $u(x,t)$ be a solution of (4.1.2) with initial data $u(x,0) = \phi(x) + \varepsilon v(x)$ with $\varepsilon \ll 1$ and $v \in L^1_N \cap L^2_N$. From Proposition 4.3.1, it follows that one may expect that the solution $u$ behaves for large time like

$$\begin{aligned} u(x,t) &\approx \phi(x) + \varepsilon e^{\mathcal{L}[\phi]t} v(x) \\ &\approx \phi(x) + \varepsilon\phi'(x)\left(\frac{1}{N}\left\langle\widetilde{\Phi}_0, v\right\rangle_{L^2_N} + s_{p,N}(t)v(x)\right) \\ &\approx \phi\left(x + \varepsilon\left(\frac{1}{N}\left\langle\widetilde{\Phi}_0, v\right\rangle_{L^2_N} + s_{p,N}(t)v(x)\right)\right), \end{aligned} \tag{4.3.10}$$

which is a space-time dependent phase modulation of the underlying periodic wave $\phi$. More precisely, note the phase modulation naturally decomposes into two parts: a spatially independent component coming from the projection of the perturbation onto the translational eigenvalue at the origin, and a space-time dependent component accounting for the dynamics associated to the accumulation of Bloch eigenvalues near the origin for large $N$. In the next section, we use this linear intuition to develop a nonlinear iteration scheme and complete the proof of Theorem 4.1.4 and its corollaries.

## 4.4 Uniform Nonlinear Asymptotic Stability

In this section, we use the decomposition of the linearized solution operator $e^{\mathcal{L}[\phi]t}$ and the associated linear estimates in Proposition 4.3.1 to develop a nonlinear iteration scheme to complete the proof of Theorem 4.1.4. As discussed at the end of Section 4.3, the linear estimates in Proposition 4.3.1 suggest that if $\phi$ is a 1-periodic diffusively spectrally stable stationary solution of (4.1.2), then $N$-periodic perturbations of $\phi$ should, for large time, behave essentially like space-time modulated version of $\phi$. This suggests a nonlinear decomposition of $N$-periodic perturbations of $\phi$, which we develop in Section 4.4.1 below. With this decomposition in hand, the proof of Theorem 4.1.4 will be completed in Section 4.4.2 through an appropriate nonlinear iteration scheme.

### 4.4.1 Nonlinear Decomposition and Perturbation Equations

Suppose $\phi$ is a 1-periodic diffusively spectrally stable stationary solution of (4.1.2). Motivated by the work in the previous section, we introduce a decomposition of nonlinear perturbations of the background wave $\phi$ which accounts for the critical phase-shift contribution $s_{p,N}(t)$ of the linear operator.

Motivated by (4.3.10), we begin by letting $\widetilde{u}(x,t)$ be a solution of (4.1.2) and define a spatially modulated function

$$u(x,t) := \widetilde{u}\left(x - \frac{1}{N}\gamma(t) - \psi(x,t), t\right) \tag{4.4.1}$$

where both $\gamma : \mathbb{R}_+ \to \mathbb{R}$ and $\psi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are functions to be determined later. Taking $\widetilde{u}$ to be initially close to $\phi$ in some sense, we attempt to decompose $u$ as

$$u(x,t) = \phi(x) + v(x,t), \tag{4.4.2}$$

where here $v$ denotes a nonlinear perturbation. Note that the form of the modulation in (4.4.1) is a combination of (i) a time-dependent modulation, as one would utilize in the proof of Proposition 4.1.3, and (ii) a space-time dependent modulation, as is used in the study of localized perturbations of periodic waves [29, 37]. Consequently, the forthcoming nonlinear analysis is essentially a mixture of these two approaches.

As a preliminary step, we derive equations that must be satisfied by the perturbation $v$ and the modulation functions $\gamma$ and $\psi$. To this end, we note that in [29, 37] it is shown through elementary, but tedious, manipulations that if $u(x,t)$ is as above then the triple $(v,\gamma,\psi)$ satisfies

$$(k\partial_t - k\mathcal{L}[\phi])\left(v + \frac{1}{N}\phi'\gamma + \phi'\psi\right) = k\widetilde{\mathcal{N}}, \quad \text{where } k\widetilde{\mathcal{N}} := \widetilde{\mathcal{Q}} + k\widetilde{\mathcal{R}}_x + k\widetilde{\mathcal{S}}_t + \widetilde{\mathcal{T}}, \tag{4.4.3}$$

with

$$\widetilde{\mathcal{Q}} := f(\phi+v) - f(\phi) - Df(\phi)v, \quad \widetilde{\mathcal{R}} := -\psi_t v - \frac{1}{N}\gamma_t v + k\left(\frac{\psi_x}{1-\psi_x}v_x\right) + k\left(\frac{\psi_x^2}{1-\psi_x}\phi'\right),$$

and

$$\widetilde{\mathcal{S}} := \psi_x v, \quad \widetilde{\mathcal{T}} := -\psi_x\left[f(\phi+v) - f(\phi)\right].$$

Rearranging slightly as in [10] to remove temporal derivatives of the perturbation $v$ in present in $\widetilde{\mathcal{N}}$ in (4.4.3) yields the following.

**Lemma 4.4.1.** *The nonlinear residual $v$ defined in* (4.4.2) *and modulation functions $\gamma$ and $\psi$ in* (4.4.1) *satisfy*

$$(k\partial_t - k\mathcal{L}[\phi])\left((1-\psi_x)v + \frac{1}{N}\phi'\gamma + \phi'\psi\right) = k\mathcal{N}, \quad \textit{where } k\mathcal{N} = \mathcal{Q} + k\mathcal{R}_x, \tag{4.4.4}$$

*where here*

$$\mathcal{Q} = (1-\psi_x)\left[f(\phi+v) - f(\phi) - Df(\phi)v\right], \tag{4.4.5}$$

*and*

$$\mathcal{R} = -\psi_t v - \frac{1}{N}\gamma_t v + c\psi_x v + k(\psi_x v)_x + k\left(\frac{\psi_x}{1-\psi_x}v_x\right) + k\left(\frac{\psi_x^2}{1-\psi_x}\phi'\right). \tag{4.4.6}$$

Our goal is to now obtain a closed nonlinear iteration scheme by integrating (4.4.4) and exploiting the decomposition of the linear solution operator $e^{\mathcal{L}[\phi]t}$ provided in (4.3.5). To motivate this, we first provide an informal description of how to determine the modulation functions $\gamma$ and $\psi$ to separate out the principle nonlinear behavior. Using Duhamel's formula, we can write (4.4.4) as the implicit integral equation

$$(1 - \psi_x(x,t))v(x,t) + \frac{1}{N}\phi'(x)\gamma(t) + \phi'(x)\psi(x,t) = e^{\mathcal{L}[\phi]t}v(x,0) + \int_0^t e^{\mathcal{L}[\phi](t-s)}\mathcal{N}(x,s)ds.$$

with initial data $\gamma(0) = 0$, $\psi(\cdot,0) = 0$ and $v(\cdot,0) = \widetilde{u}(\cdot,0) - \phi(\cdot)$. Recalling that (4.3.5) implies the linear solution operator can be decomposed as

$$e^{\mathcal{L}[\phi]t}f(x) = \phi'(x)\underbrace{\left(\frac{1}{N}\left\langle\widetilde{\Phi}_0, f\right\rangle_{L^2_N} + s_{p,N}(t)f(x)\right)}_{\text{phase modulation}} + \underbrace{\widetilde{S}(t)f(x)}_{\text{faster decaying residual}} \tag{4.4.7}$$

it follows that we can remove the principle (i.e., slowest decaying) part of the nonlinear perturbation by implicitly defining

$$\begin{cases} \gamma(t) \sim \left\langle\widetilde{\Phi}_0, v(0)\right\rangle_{L^2_N} + \int_0^t \left\langle\widetilde{\Phi}_0, \mathcal{N}(s)\right\rangle_{L^2_N} ds \\ \psi(x,t) \sim s_{p,N}(t)v(0) + \int_0^t s_{p,N}(t-s)\mathcal{N}(s)ds, \end{cases} \tag{4.4.8}$$

where here $\sim$ indicates equality for $t \geq 1$. This choice then yields the implicit description

$$v(x,t) \sim \psi_x(x,t)v(x,t) + \widetilde{S}(t)v(0) + \int_0^t \widetilde{S}(t-s)\mathcal{N}(s)ds \tag{4.4.9}$$

involving only the faster decaying residual component of the linear solution operator.

Note the above choices clearly cannot extend all the way to $t = 0$ due to an incompatibility of these choices with the initial data on $(v, \gamma, \psi)$. Here, we choose to keep the above choices for all $t \geq 1$ while interpolating between the initial data and the right hand sides of (4.4.8)-(4.4.9) on the initial layer $0 \leq t \leq 1$. Specifically, we let $\chi(t)$ be a smooth cutoff function that is zero for $t \leq 1/2$

and one for $t \geq 1$, and define the modulation functions $\gamma$ and $\psi$ implicitly for all $t \geq 0$ as

$$\begin{cases} \gamma(t) = \chi(t) \left[ \left\langle \widetilde{\Phi}_0, v(0) \right\rangle_{L^2_N} + \int_0^t \left\langle \widetilde{\Phi}_0, \mathcal{N}(s) \right\rangle_{L^2_N} ds \right] \\ \psi(x,t) = \chi(t) \left[ s_{p,N}(t) v(0) + \int_0^t s_{p,N}(t-s) \mathcal{N}(s) ds \right], \end{cases} \tag{4.4.10}$$

leaving the system

$$\begin{aligned} v(x,t) = (1-\chi(t)) &\left[ e^{\mathcal{L}[\phi]t} v(x,0) + \int_0^t e^{\mathcal{L}[\phi](t-s)} \mathcal{N}(s) ds \right] \\ &+ \chi(t) \left( \psi_x(x,t) v(x,t) + \widetilde{S}(t) v(0) + \int_0^t \widetilde{S}(t-s) \mathcal{N}(s) ds \right). \end{aligned} \tag{4.4.11}$$

We note that from the differential equation (4.4.4), along with the system of integral equations (4.4.10)-(4.4.11), we readily obtain short-time existence and continuity with respect to $t$ of a solution $(v, \psi_t, \psi_x) \in H^K_N$ and $\gamma \in W^{1,\infty}(0,\infty)$ by a standard contraction mapping argument, treating (4.4.4) as a forced heat equation: see, for example, [27]. Associated with this solution, we now aim to obtain $L^2$ estimates on $(v, \gamma_t, \psi_x, \psi_t)$ and some of their derivatives.

Noting that the nonlinear residual $\mathcal{N}$ in (4.4.4) involves only derivatives of the modulation functions $\gamma$ and $\psi$, we may then expect to extract a closed system in $(v, \gamma_t, \psi_x, \psi_t)$, and some of their derivatives, and then recover $\gamma$ and $\psi$ through the slaved system (4.4.10). In particular, observe that using (4.4.11) we see that control of $v$ in, say, $L^2_N$ requires (in part) control $v$ in $H^2_N$. This loss of derivatives is compensated by the following result, established by energy estimates in [29, 37], which uses the dissipative nature of the governing evolution equation to control higher derivatives of $v$ by lower ones, enabling us to close our nonlinear iteration.

**Proposition 4.4.2** (Nonlinear Damping)**.** *Suppose the nonlinear perturbation $v$ defined in* (4.4.2) *satisfies $v(\cdot,0) \in H^K_N$, and suppose that for some $T > 0$ the $H^K_N$ norm of $v$ and $\psi_t$, the $H^{K+1}_N$ norm of $\psi_x$, and the $L^\infty$ norms of $\gamma$ and $\gamma_t$ remain bounded by a sufficiently small constant for all* $0 \leq t \leq T$.

*Then there exist positive constants $\theta, C > 0$, both independent of $N$ and $T$, such that*

$$\|v(t)\|^2_{H^K_N} \lesssim e^{-\theta t}\|v(0)\|^2_{H^K_N} + \int_0^t e^{-\theta(t-s)}\left(\|v(s)\|^2_{L^2_N} + \|\psi_x(s)\|^2_{H^{K+1}_N} + \|\psi_t(s)\|^2_{H^K_N} + |\gamma_t(s)|^2\right) ds$$

*for all* $0 \leq t \leq T$.

*Proof.* The proof strategy is by now standard, and can be found, or example, in [29, 37]. For completeness, here we simply outline the main details. First, one rewrites (4.4.4) as the forced heat equation

$$\begin{aligned}(1-\psi_x)\left(kv_t - k^2 v_{xx}\right) = -k\left(\psi_t + \frac{1}{N}\gamma_t\right)\phi' + k^2\left(\frac{\psi_x}{1-\psi_x}\,\phi'\right)_x - \psi_x f(\phi+v) + f(\phi+v) - f(\phi)\\ + kv_x\left(c - \psi_t - \frac{\gamma_t}{N}\right) + k^2\left[\left(\frac{1}{1-\psi_x}+1\right)\psi_x v_x\right]_x.\end{aligned}$$

Multiplying by $\sum_{j=0}^K (-1)^j \frac{\partial_x^{2j} v}{1-\psi_x}$, integrating over $[0, N]$, using integration by parts and rearranging yields a bound of the form[4]

$$\begin{aligned}\partial_t\|v\|^2_{H^K_N} + 2k\|v\|^2_{H^{K+1}_N} \lesssim \varepsilon\|v\|^2_{H^{K+1}_N} + \|v\|^2_{L^2_N} + \frac{1}{\varepsilon}\left\|\frac{\psi_t}{1-\psi_x}\,\phi'\right\|^2_{H^{K-1}_N}\\ + \frac{|\gamma_t|^2}{N^2\varepsilon}\left\|\frac{1}{1-\psi_x}\,\phi'\right\|^2_{H^{K-1}_N} + \frac{1}{\varepsilon}\left\|\frac{1}{1-\psi_x}\partial_x\left(\frac{\psi_x}{1-\psi_x}\,\phi'\right)\right\|^2_{H^{K-1}_N}\\ + \frac{1}{\varepsilon}\left\|\frac{\psi_x}{1-\psi_x} f(\phi+v)\right\|^2_{H^{K-1}_N} + \frac{1}{\varepsilon}\left\|\frac{1}{1-\psi_x}\left(f(\phi+v) - f(\phi)\right)\right\|^2_{H^{K-1}_N}\\ + \frac{1}{\varepsilon}\left\|\frac{v_x}{1-\psi_x}\right\|^2_{H^{K-1}_N} + \frac{1}{\varepsilon}\left\|\frac{\psi_t v_x}{1-\psi_x}\right\|^2_{H^{K-1}_N} + \frac{|\gamma_t|^2}{N^2\varepsilon}\left\|\frac{v_x}{1-\psi_x}\right\|^2_{H^{K-1}_N}\\ + \frac{1}{\varepsilon}\left\|\frac{1}{1-\psi_x}\partial_x\left[\left(\frac{1}{1-\psi_x}+1\right)\psi_x v_x\right]\right\|^2_{H^{K-1}_N},\end{aligned}$$

where here $\varepsilon > 0$ is an arbitrary constant[5] independent of $N$. Using the Sobolev interpolation

$$\|g\|^2_{H^K_N} \leq \widetilde{C}^{-1}\|\partial_x^{K+1} g\|^2_{L^2_N} + \widetilde{C}\|g\|^2_{L^2_N},$$

[4]Below, the symbol $A \lesssim B$ implies there exists a constant $C > 0$, independent of $N$, such that $A \leq CB$.
[5]Introduced by the application of the Cauchy inequality with $\varepsilon$ throughout.

valid for some constant $\widetilde{C} > 0$ independent of $N$, now gives

$$\frac{d}{dt}\|v\|^2_{H^K_N}(t) \leq -\theta\|v(t)\|^2_{H^K_N} + C\left(\|v(t)\|^2_{L^2_N} + \|\psi_x\|^2_{H^{K+1}_N} + \|\psi_t\|^2_{H^K_N} + |\gamma_t(t)|^2\right).$$

The proof is now complete by an application of Gronwall's inequality. $\square$

### 4.4.2 Nonlinear Iteration

To complete the proof of Theorem 4.1.4, associated to the solution $(v, \gamma_t, \gamma_t, \gamma_x)$ of of (4.4.10)-(4.4.11) we define, so long as it is finite, the function

$$\zeta(t) := \sup_{0\leq s\leq t}\left(\|v(s)\|^2_{H^K_N} + \|\psi_x(s)\|^2_{H^{K+1}_N} + \|\psi_t(s)\|^2_{H^K_N} + |\gamma_t(s)|\right)^{1/2}(1+s)^{3/4}.$$

Combining the linear estimates in Proposition 4.3.1 with the damping estimate in Proposition 4.4.2, we now establish a key inequality for $\zeta$ which will yield global existence and stability of our solutions.

**Proposition 4.4.3.** *Under the assumptions of Theorem 4.1.4, there exist positive constants $C, \varepsilon > 0$, both independent of $N$, such that if $v(\cdot, 0)$ is such that*

$$E_0 := \|v(\cdot,0)\|_{L^1_N \cap H^K_N} \leq \varepsilon \quad \text{and} \quad \zeta(T) \leq \varepsilon$$

*for some $T > 0$, then we have*

$$\zeta(t) \leq C\left(E_0 + \zeta^2(t)\right)$$

*valid for all $0 \leq t \leq T$.*

*Proof.* Recalling Lemma 4.4.1 we readily see that there exists a constant $C > 0$, independent of $N$, such that

$$\|Q(t)\|_{L^1_N \cap H^1_N} \leq C\left(1 + \|\psi_x(t)\|_{H^1_N}\right)\|v(t)\|^2_{H^1_N}$$

and

$$\|\mathcal{R}(t)\|_{L^1_N\cap H^1_N} \leq C\left(\|(v, v_x, \psi_x, \psi_{xx}, \psi_t)(t)\|^2_{H^1_N} + |\gamma_t(t)|^2\right)$$

so that, using the linear estimates in Proposition 4.3.1, we have for so long as $\zeta(t)$ remains small that

$$\|Q(t)\|_{L^1_N\cap H^1_N}, \ \ \|\mathcal{R}(t)\|_{L^1_N\cap H^1_N} \leq C\zeta^2(t)(1+t)^{-3/2}$$

for some constant $C > 0$ which is independent of $N$. Since $k\mathcal{N} = Q + k\mathcal{R}_x$, it follows there exists a constant $C > 0$ independent of $N$ such that

$$\|\mathcal{N}(t)\|_{L^1_N\cap H^1_N} \leq C\|(v, v_x, v_{xx}, \psi_x, \psi_{xx}, \psi_{xxx}, \psi_t, \psi_{tx})(t)\|^2_{H^1_N} + |\gamma_t(t)|^2 \leq C\zeta^2(t)(1+t)^{-3/2}. \tag{4.4.12}$$

for so long as $\zeta(t)$ remains small. Applying the bounds in Proposition 4.3.1 to the implicit equation (4.4.11), it immediately follows that

$$\begin{aligned}
\|v(t)\|_{L^2_N} &\leq \|v(\cdot,t)\psi_x(t)\|_{L^2_N} + CE_0(1+t)^{-3/4} + C\int_0^t (1+t-s)^{-3/4}\,\|\mathcal{N}(s)\|_{L^1_N\cap L^2_N}\,ds \\
&\leq \zeta(t)^2(1+t)^{-3/2} + CE_0(1+t)^{-3/4} + C\zeta(t)^2\int_0^t (1+t-s)^{-3/4}(1+s)^{-3/2}ds \\
&\leq C\left(E_0 + \zeta(t)^2\right)(1+t)^{-3/4}
\end{aligned}$$

for some constant $C > 0$ independent of $N$. In particular, observe the loss of derivatives in the above estimate: control of the $L^2_N$ norm of $v(t)$ requires control of the $H^K_N$ norm of $v(t)$. This loss of derivatives may be compensated by the nonlinear damping estimate in Proposition 4.4.2, assuming we can obtain appropriate estimates on the modulation functions and their derivatives.

To this end, we observe that by using (4.4.10) for $0 \leq \ell \leq K+1$ we have that

$$\partial_x^\ell \psi_x(x,t) = \chi(t)\left(\partial_x^{\ell+1} s_{p,N}(t)v(0) + \int_0^t \partial_x^{\ell+1} s_{p,N}(t-s)\mathcal{N}(s)ds\right),$$

and for $0 \le \ell \le K$

$$\partial_x^\ell \psi_t(x,t) = \chi(t)\left(\partial_x^\ell \partial_t[s_{p,N}](t)v(0) + \partial_x^\ell s_{p,N}(0)\mathcal{N}(t) + \int_0^t \partial_x^\ell \partial_t[s_{p,N}](t-s)\mathcal{N}(s)ds\right)$$
$$+ \chi'(t)\left(\partial_x^\ell s_{p,N}(t)v(0) + \int_0^t \partial_x^\ell s_{p,N}(t-s)\mathcal{N}(s)ds\right),$$

and hence that

$$\|\psi_x\|_{H_N^{K+1}}, \|\psi_t\|_{H_N^K} \le C\left(E_0 + \zeta(t)^2\right)(1+t)^{-3/4}.$$

Similarly, using (4.4.10)(i) we find[6]

$$|\gamma_t(t)| = \left|\left\langle \widetilde{\Phi}_0, \mathcal{N}(t)\right\rangle_{L_N^2}\right| \le C\|\mathcal{N}(t)\|_{L_N^1} \le C\left(E_0 + \zeta^2(t)\right)(1+t)^{-3/2}.$$

Using the damping result in Proposition 4.4.2, we conclude that

$$\begin{aligned}\|v(t)\|_{H_N^K}^2 &\le CE_0^2 e^{-\theta t} + C\left(E_0 + \zeta^2(t)\right)^2 \int_0^t e^{-\theta(t-s)}(1+s)^{-3/2}ds \\ &\le CE_0^2 e^{-\theta t} + C\left(E_0 + \zeta^2(t)\right)^2 (1+t)^{-3/2} \\ &\le C\left(E_0 + \zeta^2(t)\right)^2 (1+t)^{-3/2}.\end{aligned} \tag{4.4.13}$$

Since $\zeta(t)$ is a non-decreasing function, it follows that for a given $t \in (0,T)$ we have

$$\left(\|v(s)\|_{H_N^K}^2 + \|\psi_x(s)\|_{H_N^{K+1}}^2 + \|\psi_t(s)\|_{H_N^K}^2 + |\gamma_t(s)|^2\right)^{1/2}(1+s)^{3/4} \le C\left(E_0 + \zeta^2(t)\right)^2$$

valid for all $s \in (0,t)$. Taking the supremum over $s \in (0,t)$ completes the proof. □

The proof of Theorem 4.1.4 now follows by continuous induction. Indeed, $\zeta(t)$ is continuous so long as it remains small, Proposition 4.4.3 implies that if $E_0 < \frac{1}{4C}$ then $\zeta(t) \le 2CE_0$ for all $t \ge 0$. Noting that $C > 0$ is independent of $N$, this establishes the stability estimates (4.1.7) from

[6] Note here we use an $L^\infty$-$L^1$ bound to control the inner product. This is opposed to using Cauchy-Schwartz, which would contribute the growing factor $\|\widetilde{\Phi}_0\|_{L_N^2} = \mathcal{O}(N)$.

Theorem 4.1.4 by taking

$$\widetilde{\psi}(x,t) := \frac{1}{N}\gamma(t) + \psi(x,t).$$

Further, the stability estimate (4.1.9) in Corollary 4.1.7 follows by (4.4.13) and the triangle inequality since

$$\begin{aligned}\left\| u\left(\cdot - \frac{1}{N}\gamma(t), t\right) - \phi \right\|_{L^2_N} &\leq \|u_x\|_{L^\infty} \|\psi(x,t)\|_{L^2_N} + CE_0(1+t)^{-3/4} \\ &\leq CE_0\left(1+t\right)^{-1/4},\end{aligned}$$

as claimed. Further, note that since for $0 < t < s$ we have

$$|\gamma(t) - \gamma(s)| \leq \int_t^s |\gamma_t(z)|dz \leq CE_0(1+t)^{-1/2}$$

it follows that $\gamma(t)$ converges to some[7] $\gamma_\infty \in \mathbb{R}$ as $t \to \infty$ with rate

$$|\gamma(t) - \gamma_\infty| \leq \int_t^\infty |\gamma_t(z)|dz \leq CE_0(1+t)^{-1/2},$$

which, by the triangle inequality, establishes (4.1.8), thus completing the proof of Theorem 4.1.4, as well as completes the proof of Corollary 4.1.7. In fact, notice that from (4.4.12) we have

$$\left| \int_0^t \left\langle \widetilde{\Phi}_0, \mathcal{N}(s) \right\rangle_{L^2_N} ds \right| \leq C \left\| \widetilde{\Phi}_0 \right\|_{L^\infty(\mathbb{R})} \zeta^2(t) \int_0^t (1+s)^{-3/2} ds,$$

which, since the above work shows that $\zeta(t) \leq C\varepsilon$ for some constant $C > 0$, implies from (4.4.10) that

$$\gamma_\infty = \left\langle \widetilde{\Phi}_0, v(0) \right\rangle_{L^2_N} + O(\varepsilon^2).$$

That is, the asymptotic phase shift in Theorem 4.1.4 is $O(\varepsilon^2)$ close to that suggested by the linear theory in Section 4.3.

[7]Note since the modulation function $\gamma$ depends on $N$, so does the limiting phase shift $\gamma_\infty$.

Finally, we combine Corollary 4.1.7 with Proposition 4.1.3 to establish Corollary 4.1.9. To this end, let $\varepsilon > 0$ and $C > 0$ be as in Corollary 4.1.7. Fix $N \in \mathbb{N}$ and $\delta \in (0, \delta_N)$, with $\delta_N$ as in (4.1.6), and let $\varepsilon_\delta > 0$ be as in Proposition 4.1.3. If $u_0 \in L^1_{\mathrm{per}}(0, N) \cap H^K_{\mathrm{per}}(0, N)$ with $E_0 < \varepsilon$, then Corollary 4.1.7 implies that

$$\left\| u(\cdot, t) - \phi\left(\cdot + \frac{1}{N}\gamma_\infty\right) \right\|_{H^1_{\mathrm{per}}(0,N)} \le CE_0(1+t)^{-1/4}$$

for all $t > 0$. In particular, there exists a time $T_\delta > 0$ such that

$$\left\| u(\cdot, t) - \phi\left(\cdot + \frac{1}{N}\gamma_\infty\right) \right\|_{H^1_{\mathrm{per}}(0,N)} < \varepsilon_\delta$$

for all $t \ge T_\delta$. By the translational invariance of (4.1.2) it is clear that $\phi(\cdot + \frac{1}{N}\gamma_\infty)$ is a diffusively spectrally stable 1-periodic solution of (4.1.1), and hence Proposition 4.1.3 implies[8] there exists a constant $C_\delta > 0$ such that

$$\left\| u(\cdot, t) - \phi\left(\cdot + \frac{1}{N}\gamma_\infty\right) \right\|_{H^1_{\mathrm{per}}(0,N)} \le C_\delta \varepsilon_\delta e^{-\delta t}$$

for all $t > T_\delta$. Taking $M_\delta = \frac{C_\delta \varepsilon_\delta}{E_0}$ completes the proof.

**Remark 4.4.4.** *It is expected that the analysis for the nonlinear iteration scheme in this section can be replicated with the sharper linear bounds* (4.3.6) *and* (4.3.7)*, which would allow us to attain a stability result with the appropriate Riemann sum estimates at the nonlinear level. Such a result would allow us to explicitly show both the expected exponential decay for a fixed N and the uniform polynomial decay presented in our main theorem. Further, the Riemann sum bounds make explicit the expectation that we see polynomial decay for short time and exponential decay for large time. We leave the proof of such a result for future work.*

[8]Here, we are applying Proposition 4.1.3 with initial data $u(\cdot, T_\delta)$.

# Appendix

## 4.A Bounds on Discrete Sums

In order to establish the uniform linear bounds in Proposition 4.3.1, we need to establish uniform-in-$N$ bounds on finite sums of the form

$$\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \xi^{2r}e^{-2d\xi^2 t}$$

where $N\in\mathbb{N}$. Following the ideas in [21] (see also Chapter 3), we note that the above finite sum is, up to a simple rescaling, a Riemann sum approximation for the integral

$$\int_{-\pi}^{\pi} \xi^{2r}e^{-2d\xi^2 t}d\xi$$

which, through an elementary scaling argument, exhibits $(1+t)^{-r-1/2}$ decay for large time. Using this as motivation, we now establish the following key estimate.

**Lemma 4.A.1.** *Let $d>0$ and $r\in\mathbb{N}\cup\{0\}$ be given. Then there exists a constant $C>0$, independent of $N$, such that for every $N\in\mathbb{N}$ we have*

$$\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \xi^{2r}e^{-2d\xi^2 t} \le C(1+t)^{-r-1/2},$$

*valid for all $t\ge 0$.*

*Proof.* First, consider the case when $r=0$ and note that, for each $t>0$, the function $\xi\mapsto e^{-2d\xi^2 t}$ is even and monotonically decreasing for $\xi>0$. Together with the equality $\xi_j-\xi_{j-1}=2\pi/N$, monotonicity allows us to treat the sum over $\xi\in\Omega_N$, $\xi>0$ as a right-endpoint Riemann sum (i.e. an under-approximation). Parity then tells us the sum over $\xi\in\Omega_N$, $\xi<0$ is also an under-approximation, yielding

$$\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} e^{-2d\xi^2 t} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2d\xi^2 t}\,d\xi \lesssim (1+t)^{-1/2}.$$

For $r \geq 1$, the analysis is complicated by the fact that the function

$$f(\xi,t) := \xi^{2r} e^{-2d\xi^2 t},$$

defined for $\xi \in \mathbb{R}$ and $t > 0$, is not monotonically decreasing for $\xi > 0$. However, we may use similar analysis via the following procedure.

First, observe that, for fixed $t > 0$, $f(\cdot,t)$ has a global minimum at 0 and global maxima at

$$\pm R := \pm\left(\frac{r}{2d}\right)^{1/2} t^{-1/2}, \text{ with } f(\pm R,t) = \left(\frac{r}{2de}\right)^{r} t^{-r}.$$

If $0 < t \leq r/(2d\pi^2)$, then $R \geq \pi$ so that $\pm R \notin (-\pi,\pi)$. We can then easily estimate the sum

$$\frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \xi^{2r} e^{-2d\xi^2 t} \leq \frac{1}{N}\sum_{\xi\in\Omega_N\setminus\{0\}} \pi^{2r} \leq \pi^{2r}. \tag{4.A.1}$$

For $t > r/(2d\pi^2)$, we define the auxiliary function

$$G(\xi,t) := \begin{cases} f(R,t), & |\xi| \leq R \\ f(\xi,t), & |\xi| > R \end{cases}.$$

Notice that $G(\cdot,t)$ is even and monotonically decreasing for $\xi > 0$. Furthermore, notice that

$$\int_{-\pi}^{\pi} G(\xi,t)\,d\xi \leq 2e^{1/2}\left(\frac{r}{2de}\right)^{r+1/2} t^{-r-1/2} + \int_{-\pi}^{\pi} f(\xi,t)\,d\xi \lesssim (1+t)^{-r-1/2},$$

where the last inequality follows from (4.A.1). Consequently, we may modify the monotonicity trick from the $r = 0$ case to obtain

$$\frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \xi^{2r} e^{-2d\xi^2 t} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\xi, t)\, d\xi \lesssim (1+t)^{-r-1/2}.$$

$\square$

In [21] (see also Chapter 3), the authors further established that, in the cases $r = 0$ and $r = 1$, the decay rate in Lemma 4.A.1 is indeed sharp, providing also a uniform lower bound for the corresponding finite sums. A similar analysis applied to the present situation establishes the sharpness of these bounds for all $r \geq 0$. While not necessary in the present analysis, it provides yet a deeper connection between the current uniform analysis of subharmonic perturbations and the "limiting" localized theory.

# Chapter 5

# Nonlinear Modulational Dynamics of Spectrally Stable Periodic Waves in the Lugiato-Lefever Equation

## Abstract

We consider the nonlinear stability of spectrally stable periodic stationary solutions of the Lugiato-Lefever equation (LLE), a damped nonlinear Schrödinger equation with forcing that arises in nonlinear optics. Recently there have been several works addressing the nonlinear stability of such solutions to $NT$-periodic, i.e., subharmonic, perturbations. In these works, the existence of a spectral gap yields exponential decay of sufficiently small subharmonic perturbations to a spatial shift of the underlying wave. As we have seen in the past two chapters, this result is degenerate in the limit $N = \infty$, and a stability result that is uniform in $N$ is motivated by the localized theory. Consequently, in this chapter we consider perturbations which are localized, i.e., integrable on the line, by which standard Floquet-Bloch theory implies the absence of such a spectral gap.

In Chapter 3, we used a delicate decomposition of the associated linearized solution operator to obtain linear stability results to localized perturbations with polynomial rates of decay to a spatio-temporal phase modulation of the underlying wave. However, due to the low-order damping associated to the LLE, an inherent loss of derivatives in the associated nonlinear iteration scheme prevented us from extending their linear stability result to the nonlinear level. In this chapter, we present a new nonlinear iteration scheme in which the loss of derivatives is compensated through a coupling to a separate "unmodulated" iteration scheme in which

derivatives are not lost, yet where perturbations decay too slow to close an independent iteration scheme. Our work establishes the nonlinear stability of spectrally stable periodic stationary solutions of the LLE to localized perturbations with precisely the same polynomial decay rates predicted from the linear theory. This chapter is representative of a forthcoming joint work with Mariana Haragus, Mathew Johnson, and Björn de Rijk [22].

## 5.1 Introduction

The main goal of this chapter is to return to our study of the Lugiato-Lefever equation and establish the nonlinear stability of diffusively spectrally stable periodic wave solutions to the LLE when in the presence of localized perturbations. We now summarize several important observations from the past two chapters.

In Chapter 3, we began our study of the Lugiato-Lefever equation (3.1.1)

$$\partial_t\psi = -\mathrm{i}\beta\psi_{xx} - (1+\mathrm{i}\alpha)\psi + \mathrm{i}|\psi|^2\psi + F, \tag{5.1.1}$$

an equation that has been intensely studied in he physics literature in the context of nonlinear optics; see, for example, [9] and references therein. We saw that some of the main mathematical questions raised by the physical problem concern the existence, dynamics, and stability of periodic stationary solutions. Several recent works have established the existence of periodic steady solutions $\psi(x,t) = \phi(x)$ of (5.1.1), which correspond to $T$-periodic solutions of the profile equation

$$i\beta\phi'' = -(1+i\alpha)\phi + i|\phi|^2\phi + F. \tag{5.1.2}$$

The existence of periodic solutions of (5.1.2) has been established using a variety of methods, including local bifurcation theory [11, 12, 18, 51], global bifurcation theory [49], and perturbative arguments [19].

We considered both small localized and small subharmonic perturbations of these periodic waves. As such, we let $\phi$ be a $T$-periodic steady solution[1] of (5.1.1) and decomposed $\phi = \phi_r + i\phi_i$ into its real and imaginary parts. We saw that a function of the form $\psi(x,t) = \phi(x) + v(x,t)$, with $v = v_r + iv_i$, was a solution of (5.1.1) provided that the real functions $v_r$ and $v_i$ were solutions of the real system

$$\partial_t \begin{pmatrix} v_r \\ v_i \end{pmatrix} = \mathcal{A}[\phi] \begin{pmatrix} v_r \\ v_i \end{pmatrix} + \mathcal{N}(v), \tag{5.1.3}$$

where here $\mathcal{N}(v)$ is at least quadratic in $v$ and $\mathcal{A}[\phi]$ is the real matrix differential operator

$$\mathcal{A}[\phi] = -I + \mathcal{J}\mathcal{L}[\phi], \tag{5.1.4}$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}[\phi] = \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}.$$

We considered $\mathcal{A}[\phi]$ as a linear operator with periodic coefficients acting on[2] $L^2(\mathbb{R})$ with densely defined domain $H^2(\mathbb{R})$ or as an operator acting on $L^2_{\text{per}}(0, NT)$ with densely defined domain $H^2_{\text{per}}(0, NT)$, depending on if we were considering localized or subharmonic perturbations, respectively.

In Chapter 3, we ignored the nonlinear terms and considered linear dynamics of the LLE under both subharmonic and localized perturbations. Naturally, the local dynamics about $\phi$ to both classes of perturbations is heavily influenced by the spectrum of the linearized operator $\mathcal{A}[\phi]$. Since $\mathcal{A}[\phi]$ has periodic coefficients, standard Floquet-Bloch theory implies that its spectrum as an operator acting on $L^2(\mathbb{R})$ is entirely essential and is comprised of a countable union of continuous curves which, thanks to the spatial translation invariance of (5.1.1), necessarily touches the imaginary axis at the origin. For subharmonic perturbations, the spectrum of the linear operator $\mathcal{A}[\phi]$ consists only of isolated eigenvalues with finite algebraic multiplicity. For spectrally stable waves, the spectrum

[1] A standard bootstrapping argument shows that stationary continuous solutions of the (5.1.1) are infinitely many times continuously differentiable. We therefore assume that $T$-periodic stationary solutions $\phi$ are smooth in this sense.

[2] Throughout, we will slightly abuse notation and write $L^2(\mathbb{R})$ rather than the more appropriate $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and similarly for all other Lebesgue and Sobolev spaces. Further, when the meaning is clear from context we will write functions $(f_1, f_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ as simply $f \in L^2(\mathbb{R})$.

is located in the open left half complex plane, except for $\lambda = 0$, which is an eigenvalue due to the translation invariance of (5.1.1). The presence of a spectral gap between $\lambda = 0$ and the rest of the spectrum allows in these cases to conclude on orbital nonlinear stability by standard methods. For the LLE, such a stability result has been recently obtained in [67], under the assumption that $\lambda = 0$ is a simple eigenvalue[3]; see also [51, 52, 11] for other (in)stability results. We recall that the standard orbital stability result is empty in the limit $N = \infty$ since the exponential decay rate and the allowable size of initial perturbations are both controlled by the size of the spectral gap which tends to zero as $N \to \infty$. The starting point of our analysis was the standard assumption that the $T$-periodic steady wave $\phi$ is spectrally stable in the sense of Definition 1.3.1, which we recall here for completeness.[4]

**Definition 5.1.1.** Let $T > 0$. A smooth $T$-periodic stationary solution $\phi$ of (5.1.1) is said to be *diffusively spectrally stable* provided the following conditions hold:

(i) the spectrum of the linear operator $\mathcal{A}[\phi]$ given by (5.1.4) and acting in $L^2(\mathbb{R})$ satisfies

$$\sigma(\mathcal{A}[\phi]) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\} \cup \{0\};$$

(ii) there exists $\theta > 0$ such that for any $\xi \in [-\pi/T, \pi/T)$ the real part of the spectrum of the Bloch operator $\mathcal{A}_\xi[\phi] := \mathrm{e}^{-\mathrm{i}\xi x}\mathcal{A}[\phi]\mathrm{e}^{\mathrm{i}\xi x}$ acting in $L^2_{\mathrm{per}}(0, T)$ satisfies

$$\Re\, \sigma(\mathcal{A}_\xi[\phi]) \leq -\theta\xi^2;$$

(iii) $\lambda = 0$ is a simple $T$-periodic eigenvalue of $\mathcal{A}_0[\phi]$, and the derivative $\phi'$ of the periodic wave is an associated eigenvector.

In the context of the LLE (5.1.1), the existence of diffusively spectrally stable periodic steady waves was recently established in [12]. In particular, this allows us to conclude that the set of

[3]Technically, the authors in [67] established this result only for co-periodic perturbations, $N = 1$. The extension to subharmonic perturbations with arbitrary but fixed $N > 1$, however, is straightforward.

[4]See Section 1.3.2 for further discussion of this assumption.

diffusively spectrally stable periodic steady waves of the LLE is non-empty. Relying upon a delicate decomposition of the associated linear solution operator $e^{\mathcal{A}[\phi]t}$, we established the linear stability result to localized perturbations given in Theorem 3.1.5, which we recall here for convenience.

**Theorem 5.1.2** (Localized Linear Stability, [21])**.** *Let $T > 0$ and suppose $\phi$ is a smooth $T$-periodic steady solution of* (5.1.1) *that is diffusively spectrally stable in the sense of Definition 1.3.1. Then, there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have*[5]

$$\left\| \mathrm{e}^{A[\phi]t} f \right\|_{L^2} \leq C(1+t)^{-\frac{1}{4}} \|f\|_{L^1 \cap L^2},$$

*for all $t > 0$. Furthermore, for each $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ there exists a smooth function $\gamma$ : $\mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that*

$$\|\gamma(\cdot,t)\|_{L^2} \leq C(1+t)^{-\frac{1}{4}} \|f\|_{L^1 \cap L^2},$$

*and*

$$\left\| \mathrm{e}^{A[\phi]t} f - \phi' \gamma(\cdot,t) \right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}} \|f\|_{L^1 \cap L^2},$$

*for all $t > 0$.*

**Remark 5.1.3.** *We note that the polynomial decay rates in Theorem 5.1.2 are sharp. Further, the formula for the linear phase shift $\gamma$ is completely explicit and hence the smoothness is immediatley observed; see [21] for further details. This key linear analysis will be reviewed and extended to estimates in $H^m$ in Section 5.3 below.*

The linear result Theorem 5.1.2 suggests that if $\psi$ is a solution of (5.1.1) which is initially close in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $\phi$, then for large time $\psi$ should behave approximately like

$$\psi(x,t) \approx \phi(x) + \gamma(x,t)\phi'(x) \approx \phi(x+\gamma(x,t)), \quad t \gg 1, \tag{5.1.5}$$

i.e. $\psi$ should asymptotically behave like a *spatio-temporal phase modulation* of the underlying

[5]Here and throughout the remainder of the chapter, we use the abbreviated notation $L^2$ for $L^2(\mathbb{R})$, and similarly for all other Lebesgue and Sobolev spaces.

periodic wave $\phi$. We point out that this is in stark contrast to the *standard* subharmonic theory, as it follows from the stability result in [67], where the presence of a spectral gap allowed the consideration of only temporal phase modulation functions. However, we also proved the linear stability result to subharmonic perturbations that is *uniform* in the period of the perturbations given in Theorem 3.1.3, which similarly motivated a spatio-temporal phase modulation of the underlying periodic wave $\phi$.

As we saw in Section 3.6, the introduction of spatio-temporal modulation functions leads to an inherent loss of derivatives in the nonlinear iteration scheme, and we saw that the standard way of overcoming this obstacle *fails* for the LLE. In Chapter 4, we saw that a uniform nonlinear subharmonic stability result can be derived by following the appropriate localized nonlinear iteration scheme. Consequently, the goal of this chapter is to present a new methodology to establish nonlinear staility to localized perturbations that is applicable to the LLE. The hope is that this will allow us to establish a nonlinear stability result to subharmonic perturbations which is uniform in the period of perturbation, as seen in Chapter 4.[6]

As motivated above, the main goal of this work is to upgrade the linear stability bounds in Theorem 5.1.2 to obtain a nonlinear stability result. Specifically, we establish the following main result.

**Theorem 5.1.4** (Localized Nonlinear Stability). *Let $T > 0$ and suppose $\phi$ is a smooth $T$-periodic steady solution of* (5.1.1) *that is diffusively spectrally stable in the sense of Definition 1.3.1.*[7] *Then, there exist constants $\epsilon, M > 0$ such that, whenever $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ satisfies*

$$E_0 := \|v_0\|_{L^1 \cap H^4} < \epsilon,$$

*there exist functions*

$$\widetilde{v} \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big), \quad \widetilde{v}(0) = v_0,$$

[6]This result is currently under investigation [23].

[7]This hypothesis is made throughout the whole chapter.

*and*

$$\gamma \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big), \quad \gamma(0) = 0,$$

*such that* $\psi = \phi + \widetilde{v}$ *is the unique global solution of* (5.1.1) *with initial condition* $\psi(0) = \phi + v_0$, *and the inequalities*

$$\max\left\{\|\psi(\cdot,t) - \phi\|_{L^2}, \|\gamma(\cdot,t)\|_{L^2}\right\} \leq ME_0(1+t)^{-\frac{1}{4}},$$

*and*

$$\max\left\{\|\psi\left(\cdot - \gamma(\cdot,t),t\right) - \phi\|_{L^2}, \|\partial_x\gamma(\cdot,t)\|_{H^3}, \|\partial_t\gamma(\cdot,t)\|_{H^2}\right\} \leq ME_0(1+t)^{-\frac{3}{4}},$$

*hold for all* $t \geq 0$.

**Remark 5.1.5.** *Note that the above decay rates on the nonlinear perturbations are precisely as expected from the linear theorm in Theorem 5.1.2 above and, furthermore, this provides a rigorous verification of the spatio-temporal modulated behavior* (5.1.5) *predicted from the linear theory. While the bounds stated in Theorem 5.1.4 are sharp, interestingly the proof of our main result yields additional* $L^2$ *bounds on* $\widetilde{v}_x$ *and* $\widetilde{v}_{xx}$ *which are* not *expected to be sharp. This is expected to be an artifact of our forthcoming iteration scheme. Furthermore, the need for the* $H^4$ *regularity on the initial perturbation is described in Remark 5.4.10 in Section 5.4 below. While we expect that it is possible to allow for less regular initial data in Theorem 5.1.4, we emphasize the focus of this chapter is not to obtain optimal regularity with respect to perturbations, but rather to introduce a working methodology to establish nonlinear stability of steady* $T$*-periodic waves in the LLE. Consequently, we refrain here from attempting such results.*

The proof of Theorem 5.1.4 is primarily based on an extension of the methodologies developed by Johnson et al. for the nonlinear stability analysis of periodic traveling waves in reaction diffusion systems and systems of conservation laws; see [2, 29, 37, 36, 39, 34, 31]. Those works, however, do not directly apply in this case due to a loss of derivatives in the associated nonlinear iteration scheme. Indeed, as detailed in Section 3.6 above and Section 5.4.2 below, allowing (5.1.5) to motivate the introduction of a "modulated perturbation"

$$v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x), \tag{5.1.6}$$

for some sufficiently regular function $\gamma$ we find that

$$(\partial_t - \mathcal{A}[\phi])\,(v + \gamma\phi') = F\,(v, v_x, v_{xx}, \gamma_t, \gamma_x, \gamma_{xx}, \gamma_{xxx}, \gamma_{xt})\,, \tag{5.1.7}$$

for some nonlinear function $F$ so that, in particular, attempting to control the norm of the right-hand-side of (5.1.7) in, for example, $H^1$ would require control of the perturbation $v$ in $H^3$. In the previous works by Johnson et al., this was compensated by appropriately strong damping effects in the governing evolution equation, providing a so-called "nonlinear damping estimate" which established that high Sobolev norms of solutions were exponentially slaved to low Sobolev norms, thereby allowing one to regain the lost regularity and close the iteration scheme. Specifically, this nonlinear damping technique requires that the damping effects correspond to the *highest-order spatial derivative* in the equation. In the case of the LLE equation (5.1.1), however, the damping actually corresponds to the *lowest-order derivative* which elementary calculations show negates the nonlinear damping technique leveraged in previous works. Thus, the main challenge in our proof is to replace these nonlinear damping estimates in this weakly dissipative context.

**Remark 5.1.6.** *An important observation is that in* (5.1.7) *the coefficients of the derivatives $v_x$ and $v_{xx}$ on the right hand side are multiples of $\gamma_x$ and its spatial derivatives. In particular, if one considered only time-dependent modulation functions, as is done in the case of the co-periodic perturbations [67], the equation* (5.1.7) *reduces to*

$$(\partial_t - \mathcal{A}[\phi])\,(v + \gamma\phi') = \widetilde{F}\,(v, \gamma_t) := F(v, 0, 0, \gamma_t, 0, 0, 0, 0). \tag{5.1.8}$$

*In this case, using Duhamel's principle we see that controlling the right hand side of* (5.1.8) *in $H^1$ requires control of $v$ in $H^1$, and hence the associated nonlinear iteration scheme* does not exhibit a loss of regularity in this co-periodic case*, and the same holds for subharmonic perturbations.*

To circumvent this issue, we combine the strategies in the works by Johnson et al. with a recent approach developed by Sandstede & de Rijk in [10] for the study of the stability of periodic traveling wave solutions in multi-dimensional reaction diffusion systems to perturbations which are not localized in the transverse directions to the underlying propagating wave. Their nonlinear analysis, which is based on first establishing sufficiently strong pointwise estimates on the associated linearized semigroups, relies on tracking the perturbed solution in two different coordiante systems: one of which incorporates spatio-temporal phase modulation of the underlying wave (as recommended from the linear theory), and one that does not incorporate any sort of phase modulation. Specifically, following their general approach we couple the iteration scheme for the modulated perturbation $v$ in (5.1.6) with that associated to the un-modulated perturbation

$$\widetilde{v}(x,t) = \psi(x,t) - \phi(x).$$

As we will see, while the Duhamel's principle based iteration scheme associated to $\widetilde{v}$ *does not experience a loss of regularity*, the associated decay rates of $\widetilde{v}$ are too slow to close an independent iteration scheme. Nevertheless, our work shows that the proof of Theorem 5.1.4 follows by coupling the iteration schemes for the modulated, $v$, and unmodulated, $\widetilde{v}$, perturbations, as well as using some judicious integration by parts to move derivatives off factors that may lead to a loss of derivatives whenever possible.

The outline for this chapter is as follows. In Section 5.2, we describe the spectral properties of the Bloch operators associated to $\mathcal{A}[\phi]$ and establish the existence and basic decay properties of the corresponding Bloch semigroups $e^{\mathcal{A}_\xi[\phi]t}$. In Section 5.3, we derive estimates on the evolution semigroup $e^{\mathcal{A}[\phi]t}$. Specifically, we first review the decomposition of the linear solution operator $e^{\mathcal{A}[\phi]t}$ acting on $L^2(\mathbb{R})$ that was used in the corresponding linear analysis in [21], and also extend the linear estimates obtained there from $L^2$ to $H^m$. In Section 5.3.2 we additionally provide new integration by parts formulas allowing one to move spatial derivatives between factors under the action of components of the semigroup $e^{\mathcal{A}[\phi]t}$, which provides a critical component of our strategy to avoid a loss of regularity in the nonlinaer iteration scheme. In Section 5.4, we detail the construction of our coupled iteration scheme, as well as explain our general strategy for compensating for the resulting loss of derivatives. Finally, Section 5.5 is devoted to the proof of Theorem 5.1.4, and we include in Appendix 5.A some details of the local existence and regularity theory utilized in our nonlinear analysis.

## 5.2 Spectral Preparation & Properties of Bloch Semigroups

We begin by recording spectral and semigroup properties of Bloch operators under the diffusive spectral stability assumption. In Section 3.2, we established spectral properties and semigroup estimates for the Bloch operators acting in $L^2(\mathbb{R})$. Here we extend these results to higher-regularity Sobolev spaces. This extension is necessary for the nonlinear theory.

First we note that, for functions $g \in H^m(\mathbb{R})$, we have an analogue of Parseval's equality (1.3.8) for the $H^m$-norm,[8]

$$\|g\|^2_{H^m(\mathbb{R})} \simeq \int_{-\pi/T}^{\pi/T} \|\check{g}(\xi,\cdot)\|^2_{H^m_{\mathrm{per}}(0,T)} \mathrm{d}\xi = \|\check{g}\|^2_{L^2\left([-\pi/T,\pi/T);H^m_{\mathrm{per}}(0,T)\right)}. \tag{5.2.1}$$

Also note the identity

$$\partial_x \mathcal{B}(g)(\xi,x) = \mathcal{B}\left(\partial_x g\right)(\xi,x) - \mathrm{i}\xi\check{g}(\xi,x), \tag{5.2.2}$$

which holds for $g \in H^1(\mathbb{R})$.

Recall from (3.2.1) that the Bloch operators associated with the linear operator $\mathcal{A}[\phi]$ given by (5.1.4) are defined for $\xi \in [-\pi/T, \pi/T)$ by the formula

$$\mathcal{A}_\xi[\phi] = -I + \mathcal{J}\mathcal{L}_\xi[\phi], \tag{5.2.3}$$

[8]Throughout the paper, the notation $A \lesssim B$ means that there exists a constant $C > 0$, independent of $A$ and $B$, such that $A \leq CB$, and we write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

where

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_\xi[\phi] = \begin{pmatrix} -\beta(\partial_x + i\xi)^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta(\partial_x + i\xi)^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}.$$

The $T$-periodic solution $\phi$ being smooth, the operators $\mathcal{A}_\xi[\phi]$ are closed in $L^2_{\text{per}}(0,T)$ and $H^m_{\text{per}}(0,T)$, for any $m \in \mathbb{N}$, with compactly embedded domains $H^2_{\text{per}}(0,T)$ and $H^{m+2}_{\text{per}}(0,T)$, respectively. A bootstrapping argument, just as for stationary solutions of (5.1.1), shows that eigenvectors and generalized eigenvectors of $\mathcal{A}_\xi[\phi]$ are smooth functions. As a consequence, the operators $\mathcal{A}_\xi[\phi]$ have the same spectral properties when acting in $L^2_{\text{per}}(0,T)$ or $H^m_{\text{per}}(0,T)$, and the following lemma proved in [21], see Section 3.2, for $L^2_{\text{per}}(0,T)$ remains valid for $H^m_{\text{per}}(0,T)$.

**Lemma 5.2.1** (Spectral Preparation)**.** *The Bloch operators $\mathcal{A}_\xi[\phi]$ acting in $H^m_{\text{per}}(0,T)$, for some $m \in \mathbb{N}_0$, have the following properties.*

*(i) For any fixed $\xi_0 \in (0, \pi/T)$, there exists a positive constant $\delta_0$ such that*

$$\Re\, \sigma(\mathcal{A}_\xi[\phi]) < -\delta_0,$$

*for all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.*

*(ii) There exist constants $\xi_1 \in (0, \pi/T)$ and $\delta_1 > 0$ such that for any $|\xi| < \xi_1$ the spectrum of $\mathcal{A}_\xi[\phi]$ decomposes into two disjoint subsets*

$$\sigma(\mathcal{A}_\xi[\phi]) = \sigma_-(\mathcal{A}_\xi[\phi]) \cup \sigma_0(\mathcal{A}_\xi[\phi]),$$

*with the following properties:*

*(a) $\Re\, \sigma_-(\mathcal{A}_\xi[\phi]) < -\delta_1$ and $\Re\, \sigma_0(\mathcal{A}_\xi[\phi]) > -\delta_1$;*

*(b) the set $\sigma_0(\mathcal{A}_\xi[\phi])$ consists of a single eigenvalue $\lambda_c(\xi)$ which is simple, analytic in $\xi$, and expands as*

$$\lambda_c(\xi) = \mathrm{i}a\xi - d\xi^2 + O(|\xi|^3), \tag{5.2.4}$$

*for some $a \in \mathbb{R}$ and $d > 0$;*

*(c) the eigenfunction $\Phi_\xi$ associated with $\lambda_c(\xi)$ is a smooth function, depends analytically on $\xi$, and there exists a constant $C$ such that*

$$\left\|\Phi_\xi - \phi'\right\|_{H^m_{\mathrm{per}}(0,T)} \leq C|\xi|,$$

*where $\phi'$ is the derivative of the $T$-periodic solution $\phi$;*

We point out that the expansion (5.2.4) of the simple eigenvalue $\lambda_c(\xi)$ is a consequence of the property

$$\overline{\mathcal{A}_\xi[\phi]} = \mathcal{A}_{-\xi}[\phi],$$

which holds in general for Bloch operators $\mathcal{A}_\xi$ associated to real differential operators $\mathcal{A}$. The eigenvalue $\lambda_c(\xi)$ being the only eigenvalue of $\mathcal{A}_\xi[\phi]$ with real part larger than $-\delta_1$, for all $|\xi| < \xi_1$, this implies that $\overline{\lambda_c(\xi)} = \lambda_c(-\xi)$ and then gives the expansion (5.2.4). In addition, if the periodic solution $\phi$ is an even function, which is the case for the spectrally stable periodic solutions constructed in [12], then the operator $\mathcal{A}[\phi]$ is invariant under the reflection $x \mapsto -x$, and the Bloch operators satisfy

$$R\mathcal{A}_\xi[\phi] = \mathcal{A}_{-\xi}[\phi]R, \quad (Rv)(x) = v(-x).$$

This implies that $\lambda_c(\xi) = \lambda_c(-\xi)$, which gives $a = 0$ in the expansion (5.2.4) in this case.

Finally, notice that the adjoint operator $\mathcal{A}^*_\xi[\phi]$ has similar spectral properties, its spectrum being equal to the complex conjugated spectrum of $\mathcal{A}_\xi[\phi]$. In particular, $\overline{\lambda_c(\xi)}$ is a simple eigenvalue of $\mathcal{A}^*_\xi[\phi]$ with smooth associated eigenfunction $\widetilde{\Phi}_\xi$ depending analytically on $\xi$.

Just as for the spectral properties above, the semigroup properties of the operators $\mathcal{A}_\xi[\phi]$ are the same when acting in $L^2_{\mathrm{per}}(0,T)$ or $H^m_{\mathrm{per}}(0,T)$, for any $m \in \mathbb{N}$. The following result proved in [21], see Section 3.2, for $L^2_{\mathrm{per}}(0,T)$ remains valid in $H^m_{\mathrm{per}}(0,T)$.

**Lemma 5.2.2** (Bloch semigroups)**.** *The Bloch operators $\mathcal{A}_\xi[\phi]$ acting in $H^m_{\mathrm{per}}(0,T)$, for some $m \in \mathbb{N}_0$, generate $C^0$-semigroups with the following properties.*

*(i) For any fixed $\xi_0 \in (0, \pi/T)$, there exist positive constants $C_0$ and $\mu_0$ such that*

$$\left\| e^{\mathcal{A}_\xi[\phi]t} \right\|_{\mathcal{L}(H^m_{\mathrm{per}})} \le C_0 e^{-\mu_0 t},$$

*for all $t \ge 0$ and all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.*

*(ii) With $\xi_1$ chosen as in Lemma 5.2.1 (ii), there exist positive constants $C_1$ and $\mu_1$ such that for any $|\xi| < \xi_1$, if $\Pi(\xi)$ is the spectral projection onto the (one-dimensional) eigenspace associated to the eigenvalue $\lambda_c(\xi)$ given by Lemma 5.2.1 (ii), then*

$$\left\| e^{\mathcal{A}_\xi[\phi]t} (I - \Pi(\xi)) \right\|_{\mathcal{L}(H^m_{\mathrm{per}})} \le C_1 e^{-\mu_1 t},$$

*for all $t \ge 0$.*

We refer to Section 3.2, see also [21, Section 2.3], for the proof of this result in $L^2_{\mathrm{per}}(0,T)$, which can be easily transposed to $H^m_{\mathrm{per}}(0,T)$, with $m \in \mathbb{N}$.

## 5.3 Linear Estimates

In this section, we review the decomposition of the evolution semigroup $e^{\mathcal{A}[\phi]t}$ from Section 3.3. We then derive the estimates on the evolution semigroup $e^{\mathcal{A}[\phi]t}$ needed in our subsequent nonlinear stability analysis. Note that part of the estimates established here are $H^m$-versions of the $L^2$-estimates from Section 3.3, see also [21].

### 5.3.1 Decomposition of the Evolution Semigroup

Following Section 3.3, we decompose the $C^0$-semigroup $e^{\mathcal{A}[\phi]t}$ in an exponentially decaying part and a critical part with slowest decay. In particular, recalling (3.3.1) and (3.3.4), we can decompose the $C^0$-semigroup $e^{\mathcal{A}[\phi]t}$ as

$$e^{\mathcal{A}[\phi]t} v(x) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} e^{i\xi x} e^{\mathcal{A}_\xi[\phi]t} \check{v}(\xi, x) d\xi = S_c(t)v(x) + S_e(t)v(x), \tag{5.3.1}$$

for $t \geq 0$ and $x \in \mathbb{R}$, with

$$S_c(t)v(x) := \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \rho(\xi) \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{\mathcal{A}_\xi[\phi]t} \Pi(\xi) \check{v}(\xi, x) \mathrm{d}\xi,$$

and

$$\begin{aligned} S_e(t)v(x) := \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} (1 - \rho(\xi)) \, \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{\mathcal{A}_\xi[\phi]t} \check{v}(\xi, x) \mathrm{d}\xi \\ + \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \rho(\xi) \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{\mathcal{A}_\xi[\phi]t} (1 - \Pi(\xi)) \, \check{v}(\xi, x) \mathrm{d}\xi. \end{aligned} \tag{5.3.2}$$

Recall that we take $\xi_1 \in (0, \pi/T)$ as in Lemma 5.2.2 and define a smooth nonnegative cutoff function $\rho$ satisfying $\rho(\xi) = 1$ for $|\xi| < \xi_1/2$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$. For each $|\xi| < \xi_1$, consider the spectral projection $\Pi(\xi)$ onto the one-dimensional eigenspace of $\mathcal{A}_\xi[\phi]$ associated with the eigenvalue $\lambda_c(\xi)$, given explicitly by

$$\Pi(\xi)g = \left\langle \widetilde{\Phi}_\xi, g \right\rangle_{L^2(0,T)} \Phi_\xi,$$

for any $g \in L^2(0, T)$, where $\widetilde{\Phi}_\xi$ is the smooth eigenfunction of the adjoint operator $\mathcal{A}_\xi^*[\phi]$ associated with the eigenvalue $\overline{\lambda_c(\xi)}$ satisfying $\langle \widetilde{\Phi}_\xi, \Phi_\xi \rangle_{L^2(0,T)} = 1$.

The component $S_e(t)$ from (5.3.1) is the exponentially decaying part of the evolution semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$ as the following result shows.

**Lemma 5.3.1** (Exponential decay)**.** *For any integer $m \geq 0$, there exist constants $\mu, C > 0$ such that the inequality*

$$\|S_e(t)v\|_{\mathcal{L}(H^m)} \leq C\mathrm{e}^{-\mu t},$$

*holds for any $t \geq 0$.*

*Proof.* This estimate has been established in the case $m = 0$ in [21], cf. the exponential decay estimates (3.3.2) and (3.3.5) from Section 3.3 above. The proof relies upon Parseval's equality for $L^2$-functions (1.3.8) and the $L^2$-estimates for the Bloch semigroups in Lemma 5.2.2. It is easily transferred to $m \in \mathbb{N}$ using Parseval's equality for $H^m$-functions (5.2.1) and the $H^m$-estimates for the Bloch semigroups in Lemma 5.2.2. □

To continue, we further decompose the critical component $S_c(t)$ of the semigroup in order to identify its slowest decaying component. We introduce a smooth cutoff function $\chi : [0,\infty) \to \mathbb{R}$, which vanishes on $[0,1]$ and equals 1 on $[2,\infty)$.[9] Using the explicit formula for the spectral projector $\Pi(\xi)$ and Lemma 5.2.1 we write

$$\begin{aligned}
S_c(t)v(x) &= \chi(t)S_c(t)v(x) + (1-\chi(t))S_c(t)v(x)\\
&= \frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} \Phi_\xi(x)\mathrm{d}\xi + (1-\chi(t))S_c(t)v(x)\\
&= \phi'(x)\left(\frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} \mathrm{d}\xi\right) + (1-\chi(t))S_c(t)v(x)\\
&\quad + \frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t} i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} \mathrm{d}\xi\\
&=: \phi'(x)s_p(t)v(x) + \widetilde{S}_c(t)v(x),
\end{aligned} \tag{5.3.3}$$

with

$$s_p(t)v(x) = \frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)} \mathrm{d}\xi,$$

for $t \geq 0$ and $x \in \mathbb{R}$. The motivation behind the decomposition (5.3.3) is that the $s_p(t)$-contribution corresponds precisely to the lowest $(1+t)^{-1/4}$-decay exhibited by $S_c(t)$, whereas the remaining part $\widetilde{S}_c(t)$ has faster $(1+t)^{-3/4}$-decay. The decay properties needed later in our analysis are proved in the following lemma.

**Lemma 5.3.2** (Critical Component)**.** *For all integers $\ell, j, m \geq 0$ there exist constants $C_{\ell,j}, C_m > 0$*

[9]The introduction of the smooth cutoff function $\chi$ at this point in the analysis differs from the analysis we saw in both Chapter 3 and 4. As we saw in Section 4.4.1, the reason for introducing the cutoff function $\chi$ becomes apparent only in the forthcoming nonlinear stability analysis since $\chi$ allows us to interpolate between the necessary initial conditions of the nonlinear perturbations with their necessary longtime behavior. We refer to Remark 5.4.7 for further details.

*such that*

$$\left\|\partial_x^\ell \partial_t^j s_p(t)v\right\|_{L^2} \leq C_{\ell,j}(1+t)^{-\frac{\ell+j}{2}}\|v\|_{L^2}, \quad v \in L^2(\mathbb{R}),$$
$$\left\|\partial_x^\ell \partial_t^j s_p(t)v\right\|_{L^2} \leq C_{\ell,j}(1+t)^{-\frac{1}{4}-\frac{\ell+j}{2}}\|v\|_{L^1}, \quad v \in L^1(\mathbb{R}),$$

*and*

$$\left\|\partial_x^m \widetilde{S}_c(t)v\right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}\|v\|_{L^1\cap L^2}, \quad v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

*for all* $t \geq 0$.

*Proof.* First, from Section 3.3, i.e., [21, Section 3], we have the estimates

$$\left|\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right| \lesssim \|\check{v}(\xi,\cdot)\|_{L^2(0,T)}, \qquad \left|\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right| \lesssim \|v\|_{L^1}, \tag{5.3.4}$$

Next, take $t \geq 2$, so that $\chi(t) = 1$. Then

$$\partial_x^\ell \partial_t^j s_p(t)v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \mathrm{e}^{\mathrm{i}\xi x}\rho(\xi)(i\xi)^\ell\,(\lambda_c(\xi))^j\,\mathrm{e}^{\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi,$$

and Parseval's equality (1.3.8) implies that

$$\left\|\partial_x^\ell \partial_t^j s_p(t)v\right\|_{L^2}^2 = \frac{1}{2\pi T}\int_{-\pi/T}^{\pi/T}\int_0^T \left|\rho(\xi)(\mathrm{i}\xi)^\ell\,(\lambda_c(\xi))^j\,\mathrm{e}^{\lambda_c(\xi)t}\left\langle \widetilde{\Phi}_\xi, \check{v}(\xi,\cdot)\right\rangle_{L^2(0,T)}\right|^2 \mathrm{d}x\,\mathrm{d}\xi.$$

Using (5.2.4) and (5.3.4) we find

$$\left\|\partial_x^\ell \partial_t^j s_p(t)v\right\|_{L^2} \lesssim \left\|\xi^{\ell+j}\mathrm{e}^{-\mathrm{d}\xi^2 t}\right\|_{L^\infty_\xi(\mathbb{R})}\|v\|_{L^2} \lesssim (1+t)^{-\frac{\ell+j}{2}}\|v\|_{L^2}, \quad v \in L^2(\mathbb{R}),$$
$$\left\|\partial_x^\ell \partial_t^j s_p(t)v\right\|_{L^2} \lesssim \left\|\xi^{\ell+j}\mathrm{e}^{-\mathrm{d}\xi^2 t}\right\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1} \lesssim (1+t)^{-\frac{1}{4}-\frac{\ell+j}{2}}\|v\|_{L^1}, \quad v \in L^1(\mathbb{R}),$$

which prove the first two inequalities for $t \geq 2$. Similarly, for $\widetilde{S}_c(t)$, using in addition that the quantity

$$\sup_{\xi\in[-\pi/T,\pi/T)}\left\|\partial_x^m\left(\frac{\Phi_\xi-\phi'}{i\xi}\right)\right\|_{L^\infty},$$

is finite by Lemma 5.2.1, we find

$$\left\|\partial_x^m\widetilde{S}_c(t)v\right\|_{L^2}\lesssim\left(\left\|\xi^{m+1}\mathrm{e}^{-\mathrm{d}\xi^2 t}\right\|_{L^2_\xi(\mathbb{R})}+\cdots+\left\|\xi\mathrm{e}^{-\mathrm{d}\xi^2 t}\right\|_{L^2_\xi(\mathbb{R})}\right)\|v\|_{L^1}\lesssim(1+t)^{-3/4}\|v\|_{L^1},$$

which proves the third inequality for $t\geq 2$.

The corresponding short-time bounds for $t\in[0,2]$ on $s_p(t)$ and $\widetilde{S}_c(t)$ follow similarly using that $\chi(t)$, and thus $s_p(t)$, vanishes on $[0,1]$, and that $\chi(t)$ and its derivatives are bounded on $[1,2]$. □

### 5.3.2 Integration by Parts Properties

In addition to the above linear estimates, our forthcoming nonlinear iteration scheme requires the following integration-by-parts type identities.

**Proposition 5.3.3** (Integration by Parts)**.** *Given $f,g\in H^1(\mathbb{R})$, we have the following identities for all $t\geq 0$ and $x\in\mathbb{R}$:*

$$\begin{aligned}s_p(t)\left(f\cdot\partial_x g\right)(x)&=-s_p(t)(\partial_x f\cdot g)+\partial_x s_p(t)(fg)(x)\\&\quad-\frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}\left\langle\partial_x\widetilde{\Phi}_\xi,\mathcal{B}(fg)(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi,\end{aligned}$$

*and*

$$\begin{aligned}\widetilde{S}_c(t)\left(f\cdot\partial_x g\right)(x)&=-\widetilde{S}_c(t)\left(\partial_x f\cdot g\right)\\&\quad+\frac{(1-\chi(t))}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}i\xi\Phi_\xi(x)\left\langle\widetilde{\Phi}_\xi,\mathcal{B}(fg)(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi\\&\quad-\frac{(1-\chi(t))}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}\Phi_\xi(x)\left\langle\partial_x\widetilde{\Phi}_\xi,\mathcal{B}(fg)(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi\end{aligned}$$

$$+\frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}(i\xi)^2\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\left\langle\widetilde{\Phi}_\xi,\mathcal{B}(fg)(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi$$
$$-\frac{\chi(t)}{2\pi}\int_{-\pi/T}^{\pi/T}\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}i\xi\left(\frac{\Phi_\xi(x)-\phi'(x)}{i\xi}\right)\left\langle\partial_x\widetilde{\Phi}_\xi,\mathcal{B}(fg)(\xi,\cdot)\right\rangle_{L^2(0,T)}\mathrm{d}\xi.$$

*Proof.* The proofs of the above identities are essentially the same, so we will just prove the identity for $s_p$. Using the identity (5.2.2), integration by parts gives

$$\begin{aligned}\left\langle\partial_x\widetilde{\Phi}_\xi,\mathcal{B}\left(f\cdot g\right)(\xi,\cdot)\right\rangle_{L^2(0,T)}=&-\left\langle\widetilde{\Phi}_\xi,\mathcal{B}\left(\partial_x f\cdot g+f\cdot\partial_x g\right)(\xi,\cdot)\right\rangle_{L^2(0,T)}\\&+\mathrm{i}\xi\left\langle\widetilde{\Phi}_\xi,\mathcal{B}\left(f\cdot g\right)(\xi,\cdot)\right\rangle_{L^2(0,T)}.\end{aligned}$$

Multiplying this equality by $\rho(\xi)\mathrm{e}^{\mathrm{i}\xi x+\lambda_c(\xi)t}$, integrating and rearranging terms yields the desired result. $\square$

Combining the above identities with our previous linear estimates we obtain the following result.

**Lemma 5.3.4.** *There exists a constant $C>0$, and for all integers $\ell,j\geq 0$ a constant $C_{\ell,j}>0$ such that the following inequalities hold:*

$$\begin{aligned}\left\|\partial_x^\ell\partial_t^j s_p(t)\left(f\cdot\partial_x g\right)\right\|_{L^2}&\leq C_{\ell,j}(1+t)^{-\frac{1}{4}-\frac{\ell+j}{2}}\left(\|fg\|_{L^1}+\|\partial_x f\cdot g\|_{L^1}\right),\\\left\|\widetilde{S}_c(t)\left(f\cdot\partial_x g\right)\right\|_{L^2}&\leq C(1+t)^{-\frac{3}{4}}\left(\|fg\|_{L^1}+\|\partial_x f\cdot g\|_{L^1}\right),\end{aligned}$$

*for all $f,g\in H^1(\mathbb{R})$ and $t\geq 0$, and*

$$\begin{aligned}\left\|\partial_x^\ell\partial_t^j s_p(t)\left(f\cdot\partial_x^2 g\right)\right\|_{L^2}&\leq C_{\ell,j}(1+t)^{-\frac{1}{4}-\frac{\ell+j}{2}}\left(\|fg\|_{L^1}+\|\partial_x f\cdot g\|_{L^1}+\|\partial_x^2 f\cdot g\|_{L^1}\right),\\\left\|\widetilde{S}_c(t)\left(f\cdot\partial_x^2 g\right)\right\|_{L^2}&\leq C(1+t)^{-\frac{3}{4}}\left(\|fg\|_{L^1}+\|\partial_x f\cdot g\|_{L^1}+\|\partial_x^2 f\cdot g\|_{L^1}\right),\end{aligned}$$

*for all $f,g\in H^2(\mathbb{R})$ and $t\geq 0$.*

*Proof.* The first two inequalities follow directly from the identities in Proposition 5.3.3 and the estimates in Lemma 5.3.2. For the latter ones, we use the identities

$$
\begin{aligned}
\partial_x^2 \mathcal{B}(f)(\xi, x) &= \mathcal{B}\left(\partial_x^2 f\right)(\xi, x) - 2i\xi \partial_x \mathcal{B}(f)(\xi, x) + \xi^2 \check{f}(\xi, x) \\
&= \mathcal{B}\left(\partial_x^2 f\right)(\xi, x) - 2i\xi \mathcal{B}(\partial_x f)(\xi, x) - \xi^2 \check{f}(\xi, x)
\end{aligned}
$$

to derive second-order analogues of the identities in Proposition 5.3.3. For example, we obtain

$$
\begin{aligned}
\frac{\chi(t)}{2\pi} \int_{-\pi/T}^{\pi/T} \rho(\xi) \mathrm{e}^{\mathrm{i}\xi x + \lambda_c(\xi) t} \left\langle \partial_x^2 \widetilde{\Phi}_\xi, \mathcal{B}(fg)(\xi, \cdot) \right\rangle_{L^2(0,T)} \mathrm{d}\xi &= s_p(t) \left( \partial_x^2 f \cdot g + 2\partial_x f \cdot \partial_x g + f \cdot \partial_x^2 g \right) \\
&\quad - 2\partial_x s_p(t) \left( \partial_x f \cdot g + f \cdot \partial_x g \right) + \partial_x^2 s_p(t) \left( fg \right).
\end{aligned}
$$

Then using the estimates in Lemma 5.3.2, as well as Proposition 5.3.3 to eliminate first order derivatives on $g$, yields the estimate on $s_p(t) \left( f \cdot \partial_x^2 g \right)$ and its derivatives. The estimate on $\widetilde{S}_c$ is obtained in the same way. □

**Remark 5.3.5.** *In our forthcoming analysis, we will see that Proposition 5.3.3 allows us to circumvent the loss of regularity that arises from the slowest decaying, critical part of our solution operator* $\mathrm{e}^{\mathcal{A}[\phi]t}$*. Indeed, circumventing this loss of regularity is most necessary on the critical part of our solution operator owing to the sensitivity of the algebraic decay in the nonlinear iteration scheme. However, it does not solve the loss of regularity for the exponentially decaying part of the solution operator because such an integration-by-parts type identity cannot hold for this part of the solution operator. Indeed, we can see this from the form of* (5.3.2) *and from Lemma 5.3.1. Consequently, we still experience a loss of regularity on the exponentially decaying terms, even after integration by parts. In the next section, we see that we can overcome this remaining loss of regularity after coupling the evolution of modulated perturbation to the evolution of the unmodulated perturbation.*

## 5.4 Nonlinear Iteration Scheme

The goal of this section is to introduce the nonlinear iteration scheme that will be employed in §5.5 to prove our nonlinear stability result, Theorem 5.1.4. Thus, we let $\phi$ be a smooth $T$-periodic steady wave solution of the LLE (5.1.1), which is diffusively spectrally stable, and consider the perturbed solution $\psi(t)$ of (5.1.1) with initial condition $\psi(0) = \phi + v_0$, where $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ is sufficiently small.

In Section 5.4.1 we study the nonlinear dynamics of the perturbation $\widetilde{v}(t) = \psi(t) - \phi$, and conclude that the associated linear and nonlinear estimates are too weak to close a nonlinear iteration scheme. Hence, to account for the most critical behavior (which originates from translational invariance of the steady wave $\phi$), we introduce in Section 5.4.2 a spatio-temporal phase modulation that tracks the shift of the perturbed solution in space relative to $\phi$. We establish a nonlinear iteration scheme for the modulated perturbation and the phase modulation itself. However, this scheme does not provide control over spatial derivatives of the perturbation, i.e. it exhibits a 'loss' of derivatives. We address this loss of derivative in Section 5.4.3 using integration by parts and by appending equations for the unmodulated perturbation to the scheme.

### 5.4.1 The Unmodulated Perturbation

Setting

$$\widetilde{v}(t) := \psi(t) - \phi, \tag{5.4.1}$$

the unmodulated perturbation $\widetilde{v}$ satisfies

$$(\partial_t - \mathcal{A}[\phi])\,\widetilde{v} = \widetilde{\mathcal{N}}(\widetilde{v}), \tag{5.4.2}$$

where $\mathcal{A}[\phi]$ is the linear operator from (5.1.3) and the nonlinearity $\widetilde{\mathcal{N}}$ is given by

$$\widetilde{\mathcal{N}}(\widetilde{v}) := J\left[\begin{pmatrix} 3\widetilde{v}_r^2 + \widetilde{v}_i^2 & 2\widetilde{v}_r\widetilde{v}_i \\ 2\widetilde{v}_r\widetilde{v}_i & \widetilde{v}_r^2 + 3\widetilde{v}_i^2 \end{pmatrix}\phi + |\widetilde{v}|^2\widetilde{v}\right].$$

Using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ it is straightforward to check the following estimates on the nonlinearity $\widetilde{\mathcal{N}}$.

**Lemma 5.4.1.** *For any constant $C > 0$, the inequalities*

$$\begin{aligned} \left\|\widetilde{\mathcal{N}}(\widetilde{v})\right\|_{L^1} &\lesssim \|\widetilde{v}\|_{L^2}^2 , \\ \left\|\widetilde{\mathcal{N}}(\widetilde{v})\right\|_{H^4} &\lesssim \|\widetilde{v}\|_{H^3} \|\widetilde{v}\|_{H^2} + \|\widetilde{v}\|_{H^4} \|\widetilde{v}\|_{H^1} , \end{aligned} \tag{5.4.3}$$

*hold for all $\widetilde{v} \in H^4(\mathbb{R})$ with $\|\widetilde{v}\|_{H^2} \leq C$.*

The local existence and uniqueness of the perturbation $\widetilde{v}(t)$ as a solution to (5.4.2) is an immediate consequence of the existence of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$, acting in $H^2(\mathbb{R})$, the estimates above on the nonlinearity $\widetilde{\mathcal{N}}$, and classical local existence theory for semilinear evolution problems; see, for instance, [8, Proposition 4.3.9] and [54, Theorem 6.1.3]. For completeness, the proof is contained in Appendix 5.A below.

**Proposition 5.4.2** (Local Theory for the Unmodulated Perturbation)**.** *For any $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$, there exists a maximal time $T_{\max} \in (0, \infty]$ such that* (5.4.2) *admits a unique solution*

$$\widetilde{v} \in C\big([0, T_{\max}), H^4(\mathbb{R})\big) \cap C^1\big([0, T_{\max}), H^2(\mathbb{R})\big), \tag{5.4.4}$$

*with initial condition $\widetilde{v}(0) = v_0$. In addition, if $T_{\max} < \infty$, then*

$$\lim_{t \uparrow T_{\max}} \|\widetilde{v}(t)\|_{H^2} = \infty. \tag{5.4.5}$$

Ideally, we would wish to control the perturbation $\widetilde{v}(t)$ over time, and prove that (5.4.5) cannot occur, which implies that $\widetilde{v}(t)$, and thus $\psi(t)$, are global solutions. Naively, one might expect this to be accomplished by integrating (5.4.2) and attempting to bound the perturbation $\widetilde{v}(t)$ iteratively using the Duhamel formulation

$$\widetilde{v}(t) = \mathrm{e}^{\mathcal{A}[\phi]t} v_0 + \int_0^t \mathrm{e}^{\mathcal{A}[\phi](t-s)} \widetilde{\mathcal{N}}(\widetilde{v}(s))\mathrm{d}s, \tag{5.4.6}$$

for $t \in [0, T_{\max})$. However, as outlined in Remark 5.4.3 below the temporal bounds on the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$, established in Section 5.3, are too weak to close the resulting nonlinear iteration scheme. Consequently, to utilize faster linear decay rates we introduce a spatio-temporal phase modulation $\gamma(x,t)$ for the perturbed solution $\psi$ in the next subsection, which accounts for the most critical behavior of the linear solution operator.

**Remark 5.4.3.** *The estimates on the critical component $S_c(t)$ of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$ in Lemma 5.3.2 indicate that the perturbation $\widetilde{v}(t)$ decays in $H^1(\mathbb{R})$ at most with rate $(1+t)^{-1/4}$. Thus, in an attempt to close a nonlinear iteration scheme, it makes sense to take $t > 0$ and assume that we have indeed $\|\widetilde{v}(s)\|_{H^1} \lesssim (1+s)^{-1/4}$ for $s \in [0,t)$. To close the nonlinear iteration scheme, we need to show that the right-hand side of* (5.4.6) *decays at least with rate $(1+t)^{-1/4}$. However, Lemma 5.3.2 and estimate* (5.4.3) *are insufficient to bound the contribution*

$$\int_0^t \phi' s_p(t-s)\widetilde{\mathcal{N}}(\widetilde{v}(s))\mathrm{d}s$$

*occurring on the right-hand side of* (5.4.6) *from the decomposition* (5.3.3) *of critical component $S_c(t)$ of the semigroup. Indeed, one finds the latter to be bounded by*

$$\int_0^t (1+t-s)^{-\frac{1}{4}}(1+s)^{-\frac{1}{2}}\mathrm{d}s \lesssim (1+t)^{\frac{1}{4}}.$$

*We conclude that the temporal bounds on the critical component $S_c(t)$ of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$, established in Lemmas 5.3.2 are too weak to close a nonlinear iteration scheme. This is no surprise as the same bounds on the semigroup and nonlinearity can be obtained for the nonlinear heat equation $\partial_t u = u_{xx} + u^2$ in which all nonnegative, nontrivial initial data in $H^1(\mathbb{R})$ blow up in finite time [15].*

### 5.4.2 The Modulated Perturbation

We now introduce the modulated perturbation by taking

$$v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x), \tag{5.4.7}$$

in which the spatio-temporal modulation function $\gamma(x,t)$ satisfies $\gamma(\cdot,0) = 0$, i.e., it vanishes at $t = 0$. Substituting (5.4.7) into the LLE (5.1.1), we obtain the equation

$$(\partial_t - \mathcal{A}[\phi])(v + \gamma\phi') = \mathcal{N}(v, \gamma, \partial_t\gamma) + (\partial_t - \mathcal{A}[\phi])(\gamma_x v), \tag{5.4.8}$$

where

$$\mathcal{N}(v,\gamma,\gamma_t) = \mathcal{Q}(v,\gamma) + \partial_x \mathcal{R}(v,\gamma,\gamma_t), \tag{5.4.9}$$

with

$$\mathcal{Q}(v,\gamma) = (1-\gamma_x)\, J\left[\begin{pmatrix} 3v_r^2 + v_i^2 & 2v_r v_i \\ 2v_r v_i & v_r^2 + 3v_i^2 \end{pmatrix}\phi + |v|^2 v\right],$$

and

$$\mathcal{R}(v,\gamma,\gamma_t) = -\gamma_t v - \beta J\left[\gamma_{xx} v + 2\gamma_x v_x + \frac{\gamma_x^2}{1-\gamma_x}(\phi' + v_x)\right].$$

Using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ it is straightforward to check the following estimates on the nonlinearity $\mathcal{N}$ in (5.4.8).

**Lemma 5.4.4.** *Fix a constant $C > 0$. The inequality*

$$\|\mathcal{N}(v,\gamma,\gamma_t)\|_{L^2} \lesssim \|v\|_{L^2}\|v\|_{H^1} + \|(\gamma_x,\gamma_t)\|_{H^2\times L^2}\left(\|v\|_{H^2} + \|\gamma_x\|_{L^2}\right),$$

*holds for $v \in H^2(\mathbb{R})$ and $(\gamma,\gamma_t) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})$ satisfying $\|v\|_{H^1} \le C$ and $\|\gamma\|_{H^2} \le \frac{1}{2}$.*

Integrating (5.4.8) yields the Duhamel formulation

$$v(t) + \gamma(t)\phi' = \mathrm{e}^{\mathcal{A}[\phi]t} v_0 + \int_0^t \mathrm{e}^{\mathcal{A}[\phi](t-s)} \mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s)) \mathrm{d}s + \gamma_x(t) v(t), \tag{5.4.10}$$

for $t \in [0, \tau)$, where we used the property that both $\gamma(\cdot, 0)$ and $\gamma_x(\cdot, 0)$ are identically zero. We grouped terms that are nonlinear in $v$, $\gamma$ and their derivatives on the right-hand side, whereas the left-hand side contains all contributions that are linear in $v$, $\gamma$ and their derivatives. The key idea is to make a judicious choice for $\gamma(t)$ such that the linear term $\gamma(t)\phi'$ compensates for the most critical nonlinear contributions in (5.4.10). For this, we recall from (5.3.1) and (5.3.3) that the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$ can be decomposed as

$$\mathrm{e}^{\mathcal{A}[\phi]t} = \phi' s_p(t) + \widetilde{S}(t), \tag{5.4.11}$$

with

$$\widetilde{S}(t) := \widetilde{S}_c(t) + S_e(t).$$

By Lemmas 5.3.1 and 5.3.2 the slowest temporal decay in (5.4.11) is exhibited by $\phi' s_p(t)$. This recommends the (implicit) choice

$$\gamma(t) = s_p(t) v_0 + \int_0^t s_p(t-s) \mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s)) \mathrm{d}s. \tag{5.4.12}$$

We use this equality as a definition for $\gamma$. Noting that the modulated perturbation $v$ can be written in terms of the unmodulated perturbation $\tilde{v}$ as

$$v(x,t) = \widetilde{v}(x - \gamma(x,t), t) + \phi(x - \gamma(x,t)) - \phi(x). \tag{5.4.13}$$

the equality (5.4.12) defines (implicitly) $\gamma$ as a function of the unmodulated perturbation $\widetilde{v}$. The existence and uniqueness of a local solution $\gamma$, for a given $\widetilde{v}$, is established in the following result.

**Proposition 5.4.5** (Local Theory for the Phase Modulation)**.** *For $\widetilde{v}$ given by Proposition 5.4.2,* *there exists a maximal time* $\tau_{\max} \in (0, T_{\max}]$ *such that* (5.4.12) *with* $v$ *given by* (5.4.13) *has a unique solution*

$$\gamma \in C\big([0,\tau_{\max}), H^4(\mathbb{R})\big) \cap C^1\big([0,\tau_{\max}), H^2(\mathbb{R})\big),$$

*with* $\gamma(0) = 0$*. In addition, if* $\tau_{\max} < T_{\max}$*, then*

$$\lim_{t \uparrow \tau_{\max}} \|(\gamma(t), \partial_t \gamma(t))\|_{H^4 \times H^2} = \infty.$$

We prove this proposition in Appendix 5.A. Given now the phase modulation $\gamma(t)$ in Proposition 5.4.5 and the unmodulated perturbation $\widetilde{v}(t)$ in Proposition 5.4.2, the modulated perturbation $v(t)$ is uniquely determined by (5.4.13). More precisely, we have the following result.

**Corollary 5.4.6** (The Modulated Perturbation)**.** *For* $\widetilde{v}$ *given by Proposition 5.4.2 and* $\gamma$ *given by Proposition 5.4.5, the modulated perturbation* $v$ *in* (5.4.13) *satisfies* $v \in C\big([0,\tau_{\max}), H^2(\mathbb{R})\big)$.

Substracting (5.4.12) from (5.4.10) we obtain the equation for the modulated perturbation,

$$v(t) = \widetilde{S}(t)v_0 + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s + \gamma_x(t)v(t), \tag{5.4.14}$$

which holds for $t \in [0, \tau_{\max})$. Notice that those terms exhibiting the slowest temporal decay in (5.4.10) are canceled out in (5.4.14) by our choice of $\gamma(t)$. Indeed, by Lemmas 5.3.1 and 5.3.2, the component $\widetilde{S}(t)$ of the semigroup exhibits fast decay at rate $(1+t)^{-3/4}$ instead of the $(1+t)^{-1/4}$-decay rate of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$. Moreover, the nonlinear residual $\mathcal{N}$ depends on derivatives of $\gamma$ only, which, exploiting that $s_p(0) = 0$, satisfy

$$\partial_x^\ell \partial_t^j \gamma(t) = \partial_x^\ell \partial_t^j s_p(t)v_0 + \int_0^t \partial_x^\ell \partial_t^j s_p(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\mathrm{d}s, \tag{5.4.15}$$

for $\ell, j \in \mathbb{N}_0$ and $t \in [0, \tau_{\max})$. By Lemma 5.3.2 the operators $\partial_x^\ell \partial_t^j s_p(t)$ also exhibit fast decay at rate $(1+t)^{-3/4}$ for $\ell + j \geq 1$. Therefore, one could try to close a nonlinear iteration scheme consisting of (5.4.14) and (5.4.15). This requires control over the spatial derivatives of $v$ appearing in the nonlinearity $\mathcal{N}$, which we establish in the upcoming subsection.

**Remark 5.4.7.** *The above analysis stresses the importance of the temporal cutoff function $\chi(t)$ in the decomposition* (5.3.3) *of the critical, algebraically decaying, part $S_c(t)$ of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$. Indeed, due to our choice of $\chi(t)$, the function $\gamma(t)$ is identically zero on $[0,1]$ so that the initial conditions of the modulated and unmodulated perturbation are compatible, cf.* (5.4.6) *and* (5.4.14). *In addition, taking the temporal derivative of* (5.4.12) *yields the contribution $s_p(0)\mathcal{N}(v(t), \gamma(t), \partial_t\gamma(t))$, which vanishes due to our choice of $\chi(t)$. This is crucial for obtaining sufficient regularity of $\partial_t\gamma(t)$, cf. Proposition 5.A.2. We emphasize that the introduction of the cutoff function $\chi(t)$ in* (5.3.3) *does not influence the temporal decay rates established in Lemma 5.3.2. Indeed, the decay rates of $s_p(t)$ and $\widetilde{S}_c(t)$ are determined by their behavior for large $t$ (for which $\chi(t)$ equals* 1*).*

### 5.4.3 Compensating the Loss of Derivatives

Our goal now is to close the nonlinear iteration scheme consisting of (5.4.14) and (5.4.15) by exploiting the fast temporal decay exhibited by the operators $\widetilde{S}(t)$ and $\partial_x^\ell \partial_t^j s_p(t)$ for $\ell + j \geq 1$. As discussed in the introduction, the main obstruction to closing the scheme is the lack of control over the spatial derivatives $v_x$ and $v_{xx}$ occurring in the nonlinearity $\mathcal{N}$ in (5.4.14) and (5.4.15). Naively, one would hope to control $v_x$ and $v_{xx}$ through their respective integral equations. However, simply differentiating (5.4.14), or (5.4.8), introduces third and fourth derivatives of $v$ in the nonlinearities, and thus does not resolve the issue. In addition, as mentioned in Remark 3.6.1, nonlinear damping estimates, which provide control over higher-order derivatives in terms of lower-order derivatives, are unavailable for the LLE. Instead, we address this 'loss' of derivatives by following the approach developed in [10].

The approach in [10] relies on three crucial observations. The first is that no loss of derivatives arises in the equations (5.4.2) and (5.4.6) for the unmodulated perturbation because $\widetilde{\mathcal{N}}(\widetilde{v})$ does not contain any derivatives of $\widetilde{v}$. The second is that, using the mean value theorem, the derivatives

$v_x$ and $v_{xx}$ of the modulated perturbation can be bounded in terms of the modulation $\gamma$, the unmodulated perturbation $\widetilde{v}$ and their derivatives. Hence, by *appending* the equation (5.4.6) for the unmodulated perturbation $\widetilde{v}$ to the nonlinear iteration scheme we can establish estimates on $\widetilde{v}$ and its derivatives, and thus on $v_x$ and $v_{xx}$, without 'losing' derivatives of $v$ or $\widetilde{v}$. However, the most critical behavior of the semigroup $e^{\mathcal{A}[\phi]t}$ is not factored out in (5.4.6). Consequently, the estimates on $v_x$ and $v_{xx}$ will be tame. Therefore, it is important to avoid derivatives of $v$ in the nonlinearity at crucial points. Here, the third observation comes into play: all spatial derivatives of $v$ in the nonlinearity $\mathcal{N}$ are paired with a spatial or temporal derivative of $\gamma$. Therefore, whenever possible we use the properties in Section 5.3.2 to integrate by parts and get rid of derivatives of $v$. As described in Remark 5.3.5 above, the integration by parts helps us to avoid derivatives of $v$ in the critical algebraically decaying parts of the solution operator but *not* in the exponentially decaying parts of the solution operator. As described above, we handle the loss of derivatives in $v$ and $\widetilde{v}$ by appending the evolution of the unmodulated perturbation to the evolution of the modulated perturbation.

#### 5.4.3.1 Integration by Parts

As a first step, we integrate by parts to get rid of spatial derivatives of $v$ in the algebraically decaying contributions

$$\widetilde{S}_c(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s)) \text{ and } \partial_x^\ell\partial_t^j s_p(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))$$

in (5.4.14) and (5.4.15), respectively. To this end, we decompose $\mathcal{N}$ as in (5.4.9), where $Q$ contains no derivatives of $v$ and $\partial_x\mathcal{R}$ is linear in $v$, $v_x$ and $v_{xx}$ and can be written as

$$\partial_x\mathcal{R}(v,\gamma,\gamma_t) = \mathcal{R}_1(\gamma,\gamma_t)v_{xx} + \mathcal{R}_2(\gamma,\gamma_t)v_x + \mathcal{R}_3(\gamma,\gamma_t)v + \mathcal{R}_4(\gamma).$$

Using the integration by parts formulas in Section 5.3.2, we establish the following estimates.

**Lemma 5.4.8.** *Fix a constant $C > 0$. For all integers $\ell, j$ with $0 \leq \ell, j \leq 4$, the inequalities*

$$\begin{aligned} \|Q(v,\gamma)\|_{L^1} &\lesssim \|v\|_{L^2}^2, \\ \left\|\partial_x^\ell \partial_t^j s_p(t)\left(\partial_x \mathcal{R}(v,\gamma,\gamma_t)\right)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1}{4}-\frac{\ell+j}{2}} \|(\gamma_x,\gamma_t)\|_{H^3\times H^2}\left(\|v\|_{L^2}+\|\gamma_x\|_{L^2}\right), \\ \left\|\widetilde{S}_c(t)\left(\partial_x \mathcal{R}(v,\gamma,\gamma_t)\right)\right\|_{L^2} &\lesssim (1+t)^{-\frac{3}{4}} \|(\gamma_x,\gamma_t)\|_{H^3\times H^2}\|v\|_{L^2}, \end{aligned} \tag{5.4.16}$$

*hold for $t \geq 0$, $v \in H^2(\mathbb{R})$ and $(\gamma,\gamma_t) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})$ satisfying $\|v\|_{H^1} \leq C$ and $\|\gamma\|_{H^3} \leq \frac{1}{2}$.*

*Proof.* The first inequality in (5.4.16) is an immediate consequence of the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Moreover, the same embedding also yields

$$\left\|\partial_x^\ell \mathcal{R}_k(\gamma,\gamma_t)\right\|_{L^2}, \|\mathcal{R}_4(\gamma)\|_{L^2} \lesssim \|(\gamma_x,\gamma_t)\|_{H^3\times H^2},$$

for $k = 1,2,3$, nonnegative integers $\ell$ with $k+\ell \leq 3$, and any $(\gamma,\gamma_t) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})$ satisfying $\|\gamma\|_{H^3} \leq \frac{1}{2}$. Then the last two inequalities in (5.4.16) follow from the integration by parts formulas in Lemma 5.3.4. □

Our initial goal has now been reached, as the right-hand side of (5.4.16) does not depend on any derivatives of the modulated perturbation $v$. Consequently, estimating the right hand side of, for example, (5.4.14) in $L^2$ now only requires control of the perturbation of $v$ in $L^2$, in addition to some derivatives of $\gamma_x, \gamma_t$ in $L^2$ as well.

#### 5.4.3.2 Mean Value Inequalities

In our forthcoming analysis, we need the following inequalities on the difference between the modulated and unmodulated perturbations.

**Lemma 5.4.9** (Mean Value Inequalities)**.** *For $\widetilde{v}$ and $v$ given by Proposition 5.4.2 and Corol-*

*lary 5.4.6, respectively, the inequalities*

$$\begin{aligned}
\|v(t)-\widetilde{v}(t)\|_{L^2} &\leq \left(\|\phi'\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^2}\right)\|\gamma(t)\|_{L^2},\\
\|v_x(t)-\widetilde{v}_x(t)\|_{L^2} &\leq \left(\|\phi'\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^2}\right)\|\gamma_x(t)\|_{L^2}+\left(\|\phi''\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^3}\right)\|\gamma(t)\|_{L^2},\\
\|v_{xx}(t)-\widetilde{v}_{xx}(t)\|_{L^2} &\leq \left(\|\phi'\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^2}\right)\|\gamma_{xx}(t)\|_{L^2}+2\left(\|\phi''\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^3}\right)\|\gamma_x(t)\|_{L^2}\\
&\quad+\left(\|\phi'''\|_{L^\infty}+\|\widetilde{v}(t)\|_{H^4}\right)\|\gamma(t)\|_{L^2},
\end{aligned} \tag{5.4.17}$$

*hold for all $t \in [0,\tau_{\max})$.*

*Proof.* Recall that by (5.4.7) we have

$$v(x,t)-\widetilde{v}(x,t)=\psi(x-\gamma(x,t),t)-\psi(x,t), \tag{5.4.18}$$

for $x \in \mathbb{R}$ and $t \in [0,\tau_{\max})$. By applying the mean value theorem to (5.4.18) we obtain the inequalities

$$\begin{aligned}
|v(x,t)-\widetilde{v}(x,t)| &\leq \|\psi_x(t)\|_{L^\infty}|\gamma(x,t)|,\\
|v_x(x,t)-\widetilde{v}_x(x,t)| &\leq \|\psi_x(t)\|_{L^\infty}|\gamma_x(x,t)|+\|\psi_{xx}(t)\|_{L^\infty}|\gamma(x,t)|,\\
|v_{xx}(x,t)-\widetilde{v}_{xx}(x,t)| &\leq \|\psi_x(t)\|_{L^\infty}|\gamma_{xx}(x,t)|+2\|\psi_{xx}(t)\|_{L^\infty}|\gamma_x(x,t)|\\
&\quad+\|\psi_{xx}(t)\|_{L^\infty}|\gamma_x(x,t)|^2+\|\psi_{xxx}(t)\|_{L^\infty}|\gamma(x,t)|,
\end{aligned} \tag{5.4.19}$$

for $x \in \mathbb{R}$ and $t \in [0,\tau_{\max})$. Substituting $\psi(t)=\phi+\widetilde{v}(t)$ in the above inequalities, using the smoothness of the periodic solution $\phi$ and the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, yields the result. □

These mean value inequalities connecting the unmodulated perturbation $\widetilde{v}$ to the modulated perturbation $v$ allow us to append the equation (5.4.6) for $\widetilde{v}$ to the integral system consisting of the equations (5.4.14) and (5.4.15) for $v$ and $\gamma$, and obtain a nonlinear iteration scheme in

$$\partial_x^i v(t),\ \partial_x^j\partial_t\gamma(t),\ \partial_x^\ell\gamma(t),\ \partial_x^k\widetilde{v}(t), \quad 0 \leq i,j \leq 2,\ 0 \leq k,\ell \leq 4.$$

We show in the next section that the nonlinear iteration scheme closes, which yields the proof of Theorem 5.1.4.

**Remark 5.4.10.** *The mean value inequalities* (5.4.19) *provide pointwise approximations of the spatial derivatives of the modulated perturbation* $v$ *by those of the unmodulated perturbation* $\widetilde{v}$. *Bounding the right-hand side of* (5.4.19) *requires* $L^\infty$*-estimates on the first, second and third spatial derivatives of the perturbed solution* $\psi(t) = \phi + \widetilde{v}(t)$. *Hence, any nonlinear iteration scheme exploiting the mean value inequalities* (5.4.19) *should provide control over the* $L^\infty$*-norm of* $\widetilde{v}_x, \widetilde{v}_{xx}$ *and* $\widetilde{v}_{xxx}$. *In a Hilbertian framework, as ours, such control is given by the* $H^4$*-norm of* $\widetilde{v}$, $k = 4$ *being the smallest integer for which the embedding* $H^k(\mathbb{R}) \hookrightarrow C_b^3(\mathbb{R})$ *holds. This explains the choice* $v_0 \in H^4(\mathbb{R})$ *in Theorem 5.1.4. We expect that it is possible to allow for less regular initial data in Theorem 5.1.4. However, the main purpose of this chapter is to introduce a working methodology to establish nonlinear stability of steady* $T$*-periodic waves for the LLE rather than to obtain optimal regularity with respect to perturbations.*

## 5.5 Nonlinear Stability Analysis

In this section, we establish the proof of our main result, Theorem 5.1.4, by applying the linear estimates obtained in Section 5.3 to the nonlinear iteration scheme consisting of the equations (5.4.6), (5.4.14), (5.4.15) and the inequalities (5.4.17) relating $\widetilde{v}$, $v$, $\gamma$.

*Proof of Theorem 5.1.4.* We close a nonlinear iteration scheme, controlling the unmodulated perturbation $\widetilde{v}\colon [0, T_{\max}) \to H^4(\mathbb{R})$, the phase modulation $\gamma\colon [0, \tau_{\max}) \to H^4(\mathbb{R})$ and the modulated perturbation $v\colon [0, \tau_{\max}) \to H^2(\mathbb{R})$, all defined in Section 5.4. By Propositions 5.4.2 and 5.4.5 and Corollary 5.4.6, the template function $\eta\colon [0, \tau_{\max}) \to \mathbb{R}$ given by[10]

$$\eta(t) = \sup_{0\leq s\leq t} \Big[(1+s)^{\frac{3}{4}} \left(\|v(s)\|_{L^2} + \|\partial_x\gamma(s)\|_{H^3} + \|\partial_t\gamma(s)\|_{H^2}\right) + (1+s)^{\frac{1}{4}} \left(\|\widetilde{v}(s)\|_{L^2} + \|\gamma(s)\|_{L^2}\right)$$

[10]For the motivation behind the choice of temporal weights in the template function $\eta(t)$, we refer to Remark 5.5.1 below.

$$+ (1+s)^{\frac{1}{8}} \left(\|\widetilde{v}_x(s)\|_{L^2} + \|v_x\|_{L^2}\right) + \|\widetilde{v}_{xx}(s)\|_{L^2} + \|v_{xx}(s)\|_{L^2}$$

$$\left. + (1+s)^{-\frac{1}{8}} \|\widetilde{v}_{xxx}(s)\|_{L^2} + (1+s)^{-\frac{1}{4}} \|\widetilde{v}_{xxxx}(s)\|_{L^2} \right]$$

is continuous and clearly positive and monotonically increasing. Moreover, if $\tau_{\max} < \infty$, then it holds

$$\lim_{t \uparrow \tau_{\max}} \eta(t) = \infty. \tag{5.5.1}$$

Our approach to closing the nonlinear iteration scheme is to prove that there exist constants $B > 0$ and $C > 1$ such that for all $t \in [0, \tau_{\max})$ with $\eta(t) \leq B$ we have

$$\eta(t) \leq C\left(E_0 + \eta(t)^2\right), \tag{5.5.2}$$

with $E_0$ defined in Theorem 5.1.4. Since $\eta$ is continuous, we can apply continuous induction using estimate (5.5.2). Thus, provided that $E_0 < \min\{\frac{1}{4C}, \frac{B}{2C}\}$, it follows $\eta(t) \leq 2CE_0 \leq B$, for all $t \in [0, \tau_{\max})$, which shows that (5.5.1) cannot occur. Consequently, $\tau_{\max} = \infty$ and $\eta(t) \leq 2CE_0$ for all $t \geq 0$. Upon taking $\epsilon = \min\{\frac{1}{4C}, \frac{B}{2C}\} > 0$ and $M = 2C$ this yields the result in the theorem.

It remains is to prove the key estimate (5.5.2). To this end, take $B = \frac{1}{2}$ and assume $t \in [0, \tau_{\max})$ is such that $\eta(t) \leq B$. We begin by bounding the phase modulation $\gamma(t)$ and the modulated perturbation $v(t)$ via the implicit integral equations (5.4.14) and (5.4.15), respectively. Recalling from (5.4.11) that $\widetilde{S}(t) = \widetilde{S}_c(t) + S_e(t)$, we control the contributions from the operators $\widetilde{S}_c(t)$ and $S_e(t)$ in (5.4.14) separately. To account for the $S_e(t)$ contribution in the convolution term of (5.4.14), note that Lemma 5.4.4 implies that

$$\|\mathcal{N}\left(v(s), \gamma(s), \partial_t \gamma(s)\right)\|_{L^2} \lesssim \eta(s)^2 (1+s)^{-\frac{3}{4}},$$

for $s \in [0, t]$, where we use $\eta(t) \leq B$, where here we are using that $\eta(t)$ controls the $H^2$ norm of

$v$. Hence, applying Lemma 5.3.1 we arrive at

$$\left\|\int_0^t S_e(t-s)\mathcal{N}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\mathrm{d}s\right\|_{L^2} \lesssim \int_0^t \frac{\eta(s)^2\mathrm{e}^{-\mu(t-s)}}{(1+s)^{\frac{3}{4}}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{3}{4}}}. \tag{5.5.3}$$

To control the remaining terms in (5.4.14)-(5.4.15) we note that Lemma 5.4.8 implies that

$$\begin{aligned}
\|\mathcal{Q}\left(v(s),\gamma(s)\right)\|_{L^1} &\lesssim \eta(s)^2(1+s)^{-\frac{3}{2}},\\
\left\|\partial_x^\ell\partial_t^j s_p(t-s)\left(\partial_x\mathcal{R}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\right)\right\|_{L^2} &\lesssim \eta(s)^2(1+t-s)^{-\frac{1}{4}-\frac{\ell+j}{2}}(1+s)^{-\frac{3}{2}},\\
\left\|\widetilde{S}_c(t-s)\left(\partial_x\mathcal{R}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\right)\right\|_{L^2} &\lesssim \eta(s)^2(1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{2}},
\end{aligned}$$

for $s\in[0,t]$ and $\ell,j\in\mathbb{N}_0$ with $\ell,j\leq 4$, where we use $\eta(t)\leq B$. So, applying Lemma 5.3.2 and recalling (5.4.9) we establish that

$$\begin{aligned}
\left\|\int_0^t s_p(t-s)\mathcal{N}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\mathrm{d}s\right\|_{L^2} &\lesssim \int_0^t \frac{\eta(s)^2}{(1+t-s)^{\frac{1}{4}}(1+s)^{\frac{3}{2}}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{1}{4}}},\\
\left\|\int_0^t \partial_x^\ell\partial_t^j s_p(t-s)\mathcal{N}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\mathrm{d}s\right\|_{L^2} &\lesssim \int_0^t \frac{\eta(s)^2}{(1+t-s)^{\frac{3}{4}}(1+s)^{\frac{3}{2}}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{3}{4}}},\\
\left\|\int_0^t \widetilde{S}_c(t-s)\mathcal{N}\left(v(s),\gamma(s),\partial_t\gamma(s)\right)\mathrm{d}s\right\|_{L^2} &\lesssim \int_0^t \frac{\eta(s)^2}{(1+t-s)^{\frac{3}{4}}(1+s)^{\frac{3}{2}}}\mathrm{d}s \lesssim \frac{\eta(t)^2}{(1+t)^{\frac{3}{4}}},
\end{aligned} \tag{5.5.4}$$

for $\ell,j\in\mathbb{N}_0$ with $1\leq \ell+2j\leq 4$. Thus, using Lemma 5.3.2, the decomposition (5.4.11) of $\widetilde{S}(t)$ and estimates (5.5.3) and (5.5.4), we bound the right-hand sides of (5.4.12), (5.4.14) and (5.4.15) and obtain

$$\|\gamma(t)\|_{L^2} \lesssim \frac{E_0+\eta(t)^2}{(1+t)^{\frac{1}{4}}}, \qquad \|v(t)\|_{L^2}, \|\partial_x\gamma(t)\|_{H^3}, \|\partial_t\gamma(t)\|_{H^2} \lesssim \frac{E_0+\eta(t)^2}{(1+t)^{\frac{3}{4}}}. \tag{5.5.5}$$

It remains now to provide control over the $L^2$ norms of $v_x$ and $v_{xx}$. To this end, we proceed with establishing estimates on the unmodulated perturbation $\widetilde{v}(t)$ and its derivativevs, with the goal of then using the mean value inequalities in Lemma 5.4.9 to infer control on the derivatives of $v$. An estimate on the $L^2$-norm of $\widetilde{v}(t)$ follows readily by the mean value inequalities. Indeed, combining Lemma 5.4.9 with (5.5.5) yields

$$\|\widetilde{v}(t)\|_{L^2} \lesssim \|v(t)\|_{L^2} + \left(\|\phi\|_{C_b^1} + \|\widetilde{v}(t)\|_{H^2}\right)\|\gamma(t)\|_{L^2} \lesssim \frac{E_0 + \eta(t)^2}{(1+t)^{\frac{1}{4}}} \tag{5.5.6}$$

where we use $\eta(t) \leq B$. Next, we establish a bound on the derivative $\widetilde{v}_{xxxx}(t)$. To this end, note that Lemma 5.4.1 implies that

$$\left\|\widetilde{\mathcal{N}}(\widetilde{v}(s))\right\|_{L^1} \lesssim \eta(s)^2(1+s)^{-\frac{1}{2}}, \qquad \left\|\widetilde{\mathcal{N}}(\widetilde{v}(s))\right\|_{H^4} \lesssim \eta(s)^2(1+s)^{\frac{1}{8}}, \tag{5.5.7}$$

for $s \in [0,t]$, where we use $\eta(t) \leq B$. Thus, differentiating (5.4.6) four times with respect to $x$, and using Lemma 5.3.1, the decomposition (5.3.1) of the semigroup $\mathrm{e}^{\mathcal{A}[\phi]t}$, and the estimates (5.5.7) obtain the bound

$$\begin{aligned}\|\widetilde{v}_{xxxx}(t)\|_{L^2} &\lesssim \left(\mathrm{e}^{-\mu t} + (1+t)^{-\frac{3}{4}}\right)E_0 + \int_0^t \frac{\eta(s)^2(1+s)^{\frac{1}{8}}}{\mathrm{e}^{\mu(t-s)}}\mathrm{d}s + \int_0^t \frac{\eta(s)^2}{(1+t-s)^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}\mathrm{d}s\\ &\lesssim \left(E_0 + \eta(t)^2\right)(1+t)^{\frac{1}{4}}.\end{aligned} \tag{5.5.8}$$

Using the Gagliardo-Nirenberg interpolation inequality

$$\left\|\partial_x^j \widetilde{v}(t)\right\|_{L^2} \leq C\left\|\partial_x^4 \widetilde{v}(t)\right\|_{L^2}^{j/4}\|\widetilde{v}(t)\|_{L^2}^{1-j/4}, \quad j = 1,2,3$$

for some uniform constant $C > 0$, interpolating between (5.5.6) and (5.5.8) we readily arrive at

$$\|\widetilde{v}_x(t)\|_{L^2} \lesssim \frac{E_0 + \eta(t)^2}{(1+t)^{\frac{1}{8}}}, \quad \|\widetilde{v}_{xx}(t)\|_{L^2} \lesssim E_0 + \eta(t)^2, \quad \|\widetilde{v}_{xxx}(t)\|_{L^2} \lesssim \left(E_0 + \eta(t)^2\right)(1+t)^{\frac{1}{8}}. \tag{5.5.9}$$

Subsequently, we employ the mean value inequalities in Lemma 5.4.9 to bound the derivatives of the modulated perturbation $v(t)$ in terms of derivatives of the unmodulated perturbation $\widetilde{v}(t)$.

Specifically, combining the bounds (5.4.17) with the estimates (5.5.5) and (5.5.9), we obtain

$$\begin{aligned}\|v_x(t)\|_{L^2} &\lesssim \|\widetilde{v}_x(t)\|_{L^2} + \left(\|\phi\|_{C_b^2} + \|\widetilde{v}(t)\|_{H^3}\right)\|\gamma(t)\|_{H^1} \lesssim \frac{E_0+\eta(t)^2}{(1+t)^{\frac{1}{8}}},\\ \|v_{xx}(t)\|_{L^2} &\lesssim \|\widetilde{v}_{xx}(t)\|_{L^2} + \left(\|\phi\|_{C_b^3} + \|\widetilde{v}(t)\|_{H^4}\right)\|\gamma(t)\|_{H^2} \lesssim E_0+\eta(t)^2,\end{aligned} \tag{5.5.10}$$

where we use $\eta(t) \leq B$.

Finally, by estimates (5.5.5), (5.5.6), (5.5.8), (5.5.9) and (5.5.10) it follows that there exists a constant $C > 1$, which is independent of $E_0$ and $t$, such that the key inequality (5.5.2) is satisfied, which, as discussed previously, completes the proof of Theorem 5.1.4.. □

**Remark 5.5.1.** *The choice of temporal weights in the template function $\eta(t)$ used in the proof of Theorem 5.1.4 can be motivated as follows. First, the weights applied to the terms in $\eta(t)$ involving $\|v(t)\|_{L^2}$, $\|\widetilde{v}(t)\|_{L^2}$, $\|\gamma(t)\|_{L^2}$, $\|\partial_t\gamma(t)\|_{H^2}$ and $\|\partial_x\gamma(t)\|_{H^3}$ are given by the linear theory. Indeed, Lemmas 5.3.1 and 5.3.2 imply that the linear term $\widetilde{S}(t)v_0$ in the integral equation* (5.4.14) *for $v(t)$ exhibits $(1+t)^{-3/4}$-decay, whereas the linear terms $\mathrm{e}^{\mathcal{A}[\phi]t}v_0$ and $\partial_x^\ell\partial_t^j s_p(t)$ in the integral equations* (5.4.6) *and* (5.4.15) *for $\widetilde{v}(t)$ and $\gamma(t)$ exhibit decay at rates $(1+t)^{-1/4}$ and $(1+t)^{-1/4-(\ell+j)/2}$, respectively. Next, the temporal weight applied to the contribution $\|\widetilde{v}_{xxxx}(t)\|_{L^2}$ in $\eta(t)$ arises by bounding the most critical nonlinear term in the integral equation* (5.4.6) *for $\widetilde{v}(t)$, which, as outlined in Remark 5.4.3, grows at rate $(1+t)^{1/4}$. Finally, the weights applied to $\|\widetilde{v}_x(t)\|_{L^2}$, $\|\widetilde{v}_{xx}(t)\|_{L^2}$ and $\|\widetilde{v}_{xxx}(t)\|_{L^2}$ arise by interpolation, whereas the weights applied to $\|v_x(t)\|_{L^2}$ and $\|v_{xx}(t)\|_{L^2}$ are directly linked to those applied to $\|\widetilde{v}_x(t)\|_{L^2}$ and $\|\widetilde{v}_{xx}(t)\|_{L^2}$.*

# Appendix

## 5.A Local Theory

In this appendix we collect the necessary local theory for our nonlinear stability analysis. We first establish local existence and uniqueness of the unmodulated perturbation.

*Proof of Proposition 5.4.2.* We note that (5.4.2) can be written as the semilinear evolution problem

$$\partial_t \widetilde{v} = L\widetilde{v} + N(\widetilde{v}),$$

where $L\colon D(L) \subset H^2(\mathbb{R}) \to H^2(\mathbb{R})$, with $D(L) = H^4(\mathbb{R})$ and $L\widetilde{v} = -J\beta\partial_{xx}\widetilde{v}$, is a densely defined skew-adjoint linear operator, and $N\colon H^2(\mathbb{R}) \to H^2(\mathbb{R})$ given by

$$N(\widetilde{v}) = \widetilde{\mathcal{N}}(\widetilde{v}) + \mathcal{A}[\phi]\widetilde{v} - L\widetilde{v},$$

is locally Lipschitz. Hence, $L$ is $m$-dissipative, and thus generates a $C^0$-semigroup by the Lumer-Phillips theorem. Thus, noting that the Hilbert space $H^2(\mathbb{R})$ is reflexive, the result follows immediately from classical local existence theory for semilinear evolution problems, cf. [54, Theorem 6.1.3] and [8, Proposition 4.3.9]. □

Next, we establish local existence and uniqueness of the spatio-temporal phase modulation $\gamma$, which arises as the solution of the integral equation (5.4.12). In particular, the result in Proposition 5.4.5 is a consequence of the result in Proposition 5.A.2 below. First, we prove the following preliminary result.

**Lemma 5.A.1.** *For $\widetilde{v}$ given by Proposition 5.4.2, the mapping $V\colon H^2(\mathbb{R}) \times [0, T_{\max}) \to H^2(\mathbb{R})$,*

$$V(\gamma, t)[x] = \widetilde{v}(x - \gamma(x), t) + \phi(x - \gamma(x)) - \phi(x),$$

*is well-defined, continuous in $t$, and locally Lipschitz continuous in $\gamma$ (uniformly in $t$ on compact subintervals of $[0, T_{\max})$).*

*Proof.* First, we note the embedding $H^4(\mathbb{R}) \hookrightarrow C_b^3(\mathbb{R})$ implies that

$$\widetilde{v} \in C\big([0, T_{\max}), C_b^3(\mathbb{R})\big). \tag{5.A.1}$$

Therefore, the mean value theorem yields

$$\|V(\gamma_1, t) - V(\gamma_2, t)\|_{H^2} \lesssim \|\widetilde{v}(t) + \phi\|_{C_b^3} \|\gamma_1 - \gamma_2\|_{H^2}, \tag{5.A.2}$$

for $\gamma_{1,2} \in H^2(\mathbb{R})$ and $t \in [0, T_{\max})$. Taking $\gamma_2 = 0$ in (5.A.2) and noting that $V(0, t) = \widetilde{v}(t) \in H^2(\mathbb{R})$, shows that $V$ is well-defined. Moreover, (5.A.1) and (5.A.2) yield Lipschitz continuity of $V$ in $\gamma$ (uniformly in $t$ on compact subintervals of $[0, T_{\max})$).

Similarly as in (5.A.2), we employ the mean value theorem and (5.A.1) to obtain

$$\begin{aligned}\|V(\gamma, t) - V(\gamma, s)\|_{H^2} &\lesssim \|(V(\gamma, t) - V(\gamma, s)) - (V(0, t) - V(0, s))\|_{H^2} + \|V(0, t) - V(0, s)\|_{H^2} \\ &\lesssim \|\widetilde{v}(t) - \widetilde{v}(s)\|_{C_b^3} \|\gamma\|_{H^2} + \|\widetilde{v}(t) - \widetilde{v}(s)\|_{H^2},\end{aligned}$$

for $\gamma \in H^2(\mathbb{R})$ and $s, t \in [0, T_{\max})$. Continuity of $V$ with respect to $t$ now follows by (5.4.4) and (5.A.1). $\square$

**Proposition 5.A.2.** *For $\widetilde{v}$ given by Proposition 5.4.2, let $V\colon H^2(\mathbb{R}) \times [0, T_{\max}) \to H^2(\mathbb{R})$ be the mapping in Lemma 5.A.1. Then, there exists a maximal time $\tau_{\max} \in (0, T_{\max}]$ such that the integral* *system*

$$\begin{aligned}\gamma(t) &= s_p(t)v_0 + \int_0^t s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s))\mathrm{d}s,\\ \gamma_t(t) &= \partial_t s_p(t)v_0 + \int_0^t \partial_t s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s))\mathrm{d}s,\end{aligned} \tag{5.A.3}$$

*has a unique solution*

$$(\gamma,\gamma_t) \in C\big([0,\tau_{\max}), H^4(\mathbb{R}) \times H^2(\mathbb{R})\big).$$

*In addition, if* $\tau_{\max} < T_{\max}$*, then*

$$\lim_{t\uparrow\tau_{\max}} \|(\gamma,\gamma_t)\|_{H^4\times H^2} = \infty, \tag{5.A.4}$$

*holds. Finally,* $\gamma \in C^1\big([0,\tau_{\max}), H^2(\mathbb{R})\big)$ *and* $\partial_t\gamma(t) = \gamma_t(t)$ *for* $t \in [0,\tau_{\max})$.

*Proof.* First, the result in Lemma 5.3.2 implies that the operators $s_p(t)\colon L^2(\mathbb{R}) \to H^4(\mathbb{R})$ and $\partial_t s_p(t)\colon L^2(\mathbb{R}) \to H^2(\mathbb{R})$ are $t$-uniformly bounded and strongly continuous on $[0,\infty)$. Next, recall that the nonlinearity $\mathcal{N}$ can be decomposed as in (5.4.9), where $\mathcal{Q}$ contains no derivatives of $v$ and $\partial_x\mathcal{R}$ is linear in $v$, $v_x$ and $v_{xx}$. Then, it follows from Lemmas 5.4.4 and 5.A.1 that the nonlinear map $N\colon H^4(\mathbb{R}) \times H^2(\mathbb{R}) \times [0,T_{\max}) \to L^2(\mathbb{R})$ given by

$$N(\gamma,\gamma_t,t) = \mathcal{N}(V(\gamma,t),\gamma,\gamma_t),$$

is well-defined, continuous in $t$, and locally Lipschitz continuous in $(\gamma,\gamma_t)$ (uniformly in $t$ on compact subintervals of $[0,\tau_{\max})$), where we also used the inequalities

$$\left\|\partial_x^\ell f \cdot \partial_x^k g\right\|_{L^2} \leq \|f\|_{H^2}\|g\|_{H^4}, \qquad 0 \leq k \leq 3,\ 0 \leq l \leq 2,$$

to bound the $L^2$-norm of products for functions $f \in H^2(\mathbb{R})$ and $g \in H^4(\mathbb{R})$.

Standard arguments, see for instance [8, Proposition 4.3.3] or [54, Theorem 6.1.4], now imply that there exist constants $R > 0$ and $\tau \in (0, T_{\max})$ such that $\Psi\colon C\big([0,\tau], B(R)\big) \to C\big([0,\tau], B(R)\big)$ given by

$$\Psi(\gamma,\gamma_t)[t] = \begin{pmatrix} s_p(t)v_0 \\ \partial_t s_p(t)v_0 \end{pmatrix} + \int_0^t \begin{pmatrix} s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s)) \\ \partial_t s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s)) \end{pmatrix} \mathrm{d}s,$$

is a well-defined contraction mapping, where $B(R)$ is the closed ball centered at the origin in $H^4(\mathbb{R}) \times H^2(\mathbb{R})$ of radius $R$. Hence, by the Banach fixed point theorem, $\Psi$ admits a unique fixed point, which yields a unique solution $(\gamma,\gamma_t) \in C\big([0,\tau], H^4(\mathbb{R}) \times H^2(\mathbb{R})\big)$ to (5.A.3). Letting $\tau_{\max} \in (0, T_{\max}]$ be the supremum of all such $\tau$, we obtain a maximally defined solution $(\gamma,\gamma_t) \in C\big([0,\tau_{\max}), H^4(\mathbb{R}) \times H^2(\mathbb{R})\big)$ to (5.A.3).

Next, assume by contradiction that $\tau_{\max} < T_{\max}$ and (5.A.4) does not hold. Take $t_0 \in [0,\tau_{\max})$. Similarly as before, one proves that there exist constants $M, \delta > 0$, which are independent of $t_0$, such that $\Psi_{t_0}\colon C\big([t_0, t_0+\delta], B(M)\big) \to C\big([t_0, t_0+\delta], B(M)\big)$ given by

$$\begin{aligned}\Psi_{t_0}(\widetilde{\gamma},\widetilde{\gamma}_t)[t] = &\begin{pmatrix} s_p(t)v_0 \\ \partial_t s_p(t)v_0 \end{pmatrix} + \int_0^{t_0} \begin{pmatrix} s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s)) \\ \partial_t s_p(t-s)\mathcal{N}(V(\gamma(s),s),\gamma(s),\gamma_t(s)) \end{pmatrix} \mathrm{d}s \\ &+ \int_{t_0}^t \begin{pmatrix} s_p(t-s)\mathcal{N}(V(\widetilde{\gamma}(s),s),\widetilde{\gamma}(s),\widetilde{\gamma}_t(s)) \\ \partial_t s_p(t-s)\mathcal{N}(V(\widetilde{\gamma}(s),s),\widetilde{\gamma}(s),\widetilde{\gamma}_t(s)) \end{pmatrix} \mathrm{d}s,\end{aligned}$$

is a well-defined contraction mapping, which admits a unique fixed point $(\widetilde{\gamma},\widetilde{\gamma}_t) \in C\big([t_0, t_0 + \delta], H^4(\mathbb{R}) \times H^2(\mathbb{R})\big)$. Setting $t_0 := \tau_{\max} - \delta/2$, it readily follows that $(\check{\gamma},\check{\gamma}_t) \in C\big([0, \tau_{\max} + \delta/2], H^4(\mathbb{R}) \times H^2(\mathbb{R})\big)$ given by

$$(\check{\gamma}(t),\check{\gamma}(t)) = \begin{cases} (\gamma(t),\gamma_t(t)), & t \in [0, \tau_{\max} - \frac{\delta}{2}], \\ (\widetilde{\gamma}(t),\widetilde{\gamma}_t(t)), & t \in [\tau_{\max} - \frac{\delta}{2}, \tau_{\max} + \frac{\delta}{2}], \end{cases}$$

solves (5.A.3), which contradicts the maximality of $\tau_{\max}$. We conclude that if $\tau_{\max} < T_{\max}$, then (5.A.4) must hold.

Finally, Lemma 5.3.2 readily implies that $\gamma(t)$ is differentiable on $[0, \tau_{\max})$ with $\partial_t \gamma = \gamma_t$, where we use $s_p(0) = 0$. This completes the proof. $\square$

# References


[1] M. J. Ablowitz, , and M. A. Hoefer. Dispersive shock waves. *Scholarpedia*, 4(11):5562, 2009.

[2] B. Barker, M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation. *Phys. D*, 258:11–46, 2013.

[3] S. Benzoni-Gavage, P. Noble, and L. M. Rodrigues. Slow modulations of periodic waves in Hamiltonian PDEs, with application to capillary fluids. *J. Nonlinear Sci.*, 24(4):711–768, 2014.

[4] N. Boussaid and L. Delcey. Asymptotic stability of periodic solutions of the Lugiato-Lefever equation. *In preparation*, 2020.

[5] J. C. Bronski, V. M. Hur, and M. A. Johnson. Modulational instability in equations of KdV type. In *New approaches to nonlinear waves*, volume 908 of *Lecture Notes in Phys.*, pages 83–133. Springer, Cham, 2016.

[6] J. C. Bronski and M. A. Johnson. The modulational instability for a generalized Korteweg-de Vries equation. *Arch. Ration. Mech. Anal.*, 197(2):357–400, 2010.

[7] J. C. Bronski, M. A. Johnson, and T. Kapitula. An index theorem for the stability of periodic traveling waves of kdv type. *Proceedings of the Royal Society of Edinburgh: Section A*, 141(6):1141–1173, 2011.

[8] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[9] Y. K. Chembo, D. Gomila, M. Tlidi, and C. R. Menyuk. Topical issue: theory and applications of the Lugiato-Lefever equation. *Eur. Phys. J. D*, 71, 2017.

[10] B. de Rijk and B. Sandstede. Diffusive stability against nonlocalized perturbations of planar wave trains in reaction-diffusion systems. *Journal of Differential Equations*, 274:1223–1261, 2021.

[11] L. Delcey and M. Haragus. Instabilities of periodic waves for the Lugiato-Lefever equation. *Rev. Roumaine Math. Pures Appl.*, 63(4):377–399, 2018.

[12] L. Delcey and M. Haragus. Periodic waves of the Lugiato-Lefever equation at the onset of Turing instability. *Philos. Trans. Roy. Soc. A*, 376(2117):20170188, 21, 2018.

[13] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider. The dynamics of modulated wave trains. *Mem. Amer. Math. Soc.*, 199(934):viii+105, 2009.

[14] T. Elperin, N. Kleeorin, and A. Krylov. Nondissipative shock waves in two-phase flows. *Physica D*, 74, 1994.

[15] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[16] H. Gaebler and M. Stanislavova. NLS and KdV Hamiltonian linearized operators: A priori bounds on the spectrum and optimal $L^2$ estimates for the semigroups. *Physica D*, 416, 2021.

[17] R. A. Gardner. On the structure of the spectra of periodic travelling waves. *J. Math. Pures Appl. (9)*, 72(5):415–439, 1993.

[18] C. Godey. A bifurcation analysis for the Lugiato-Lefever equation. *Eur. Phys. J. D*, 71:131, 2017.

[19] S. Hakkaev, M. Stanislavova, and A. G. Stefanov. On the generation of stable Kerr frequency combs in the Lugiato-Lefever model of periodic optical waveguides. *SIAM J. Appl. Math.*, 79(2):477–505, 2019.

[20] M. Haragus. Stability of periodic waves for the generalized BBM equation. *Rev. Roumaine Maths. Pures Appl.*, 53(5-6):445–463, 2008.

[21] M. Haragus, M. A. Johnson, and W. R. Perkins. Linear modulational and subharmonic dynamics of spectrally stable Lugiato-Lefever periodic waves. *Journal of Differential Equations*, 280:315–354, 2021.

[22] M. Haragus, M. A. Johnson, W. R. Perkins, and B. de Rijk. Nonlinear stability of spectrally stable Lugiato-Lefever periodic waves. Part I: Localized perturbations. *In preparation*, 2021.

[23] M. Haragus, M. A. Johnson, W. R. Perkins, and B. de Rijk. Nonlinear stability of spectrally stable Lugiato-Lefever periodic waves. Part II: Subharmonic perturbations. *In preparation*, 2021.

[24] S. E. Harris. Conservation laws for a nonlinear wave equation. *Nonlinearity*, 9:187–208, 1996.

[25] S. E. Harris and P. A. Clarkson. Painleve analysis and similarity reductions for the magma equations. *SIGMA*, 68, 2006.

[26] K. R. Helfrich and J. A. Whitehead. Solitary waves on conduits of bouyant fluid in a more viscous fluid. *Geophys Astro Fluid*, 51:35–52, 1990.

[27] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[28] M. A. Johnson. Stability of small periodic waves in fractional KdV-type equations. *SIAM J. Math. Anal.*, 45(5):3168–3193, 2013.

[29] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability. *Arch. Ration. Mech. Anal.*, 207(2):693–715, 2013.

[30] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: the Whitham equation. *Arch. Ration. Mech. Anal.*, 207(2):669–692, 2013.

[31] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations. *Inventiones Mathematicae*, 197(1):115–213, 2014.

[32] M. A. Johnson and W. R. Perkins. Modulational instability of viscous fluid conduit periodic waves. *SIAM Journal on Mathematical Analysis*, 52(1):277–305, 2020.

[33] M. A. Johnson and W. R. Perkins. Subharmonic dynamics of wave trains in reaction–diffusion systems. *Physica D: Nonlinear Phenomena*, 422:132891, 2021.

[34] M. A. Johnson and K. Zumbrun. Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case. *J. Differential Equations*, 249(5):1213–1240, 2010.

[35] M. A. Johnson and K. Zumbrun. Rigorous justification of the whitham modulation equations for the generalized Korteweg–de Vries equation. *Studies in Applied Mathematics*, 125(1):69–89, 2010.

[36] M. A. Johnson and K. Zumbrun. Nonlinear stability of periodic traveling-wave solutions of viscous conservation laws in dimensions one and two. *SIAM J. Appl. Dyn. Syst.*, 10(1):189–211, 2011.

[37] M. A. Johnson and K. Zumbrun. Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction-diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(4):471–483, 2011.

[38] M. A. Johnson, K. Zumbrun, and J. C. Bronski. On the modulation equations and stability of

periodic generalized Korteweg-de Vries waves via Bloch decompositions. *Phys. D*, 239(23-24):2057–2065, 2010.

[39] M. A. Johnson, K. Zumbrun, and P. Noble. Nonlinear stability of viscous roll waves. *SIAM J. Math. Anal.*, 43(2):577–611, 2011.

[40] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*, volume 185 of *Applied Mathematical Sciences*. Springer, New York, 2013. With a foreword by Christopher K. R. T. Jones.

[41] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 2nd edition, 1976.

[42] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 1995.

[43] N. K. Lowman and M. A. Hoefer. Dispersive shock waves in viscously deformable media. *Journal of Fluid Mechanics*, 718:524–557, 2013.

[44] N. K. Lowman, M. A. Hoefer, and G. A. El. Interactions of large amplitude solitary waves in viscous fluid conduits. *J. Fluid Mech.*, 750:372–384, 2014.

[45] L. A. Lugiato and R. Lefever. Spatial dissipative structures in passive optical systems. *Phys. Rev. Lett.*, 58, 1987.

[46] J. C. Luke. A perturbation method for nonlinear dispersive wave problems. *Proc. R. Soc. Lond. A*, 292:402–412, 1966.

[47] M. D. Maiden and M. A. Hoefer. Modulations of viscous fluid conduit periodic waves. *Proceedings of the Royal Society A.*, 472, 2016.

[48] M. D. Maiden, N. K. Lowman, D. V. Anderson, M. E. Schubert, and M. A. Hoefer. Observation of dispersive shock waves, solitons, and their interactions in viscous fluid conduits. *Physical Review Letters*, 116(174501), 2016.

[49] R. Mandel and W. Reichel. A priori bounds and global bifurcation results for frequency combs modeled by the Lugiato-Lefever equation. *SIAM J. Appl. Math.*, 77(1):315–345, 2017.

[50] T. R. Marchant and N. F. Smyth. Approximate solutions for dispersive shock waves in nonlinear media. *J. Nonlinear optic. Phys. Mat.*, 21(18), 2012.

[51] T. Miyaji, I. Ohnishi, and Y. Tsutsumi. Bifurcation analysis to the Lugiato-Lefever equation in one space dimension. *Phys. D*, 239(23-24):2066–2083, 2010.

[52] T. Miyaji, I. Ohnishi, and Y. Tsutsumi. Stability of a stationary solution for the Lugiato-Lefever equation. *Tohoku Math. J. (2)*, 63(4):651–663, 2011.

[53] P. Olson and U. Christensen. Solitary wave propagation in a fluid conduit within a viscous matrix. *Journal of Geophysical Research*, 91(B6):6367 – 6374, 1986.

[54] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[55] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[56] B. Sandstede, A. Scheel, G. Schneider, and H. Uecker. Diffusive mixing of periodic wave trains in reaction-diffusion systems. *J. Differential Equations*, 252(5):3541–3574, 2012.

[57] A. Scheel and Q. Wu. Diffusive stability of Turing patterns via normal forms. *J. Dynam. Differential Equations*, 27(3-4):1027–1076, 2015.

[58] G. Schneider. Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation. *Comm. Math. Phys.*, 178(3):679–702, 1996.

[59] G. Schneider. Nonlinear diffusive stability of spatially periodic solutions—abstract theorem and higher space dimensions. In *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997)*, volume 8 of *Tohoku Math. Publ.*, pages 159–167. Tohoku Univ., Sendai, 1998.

[60] G. Schneider. Nonlinear stability of Taylor vortices in infinite cylinders. *Arch. Rational Mech. Anal.*, 144(2):121–200, 1998.

[61] D. R. Scott and D. J. Stevenson. Magma solitons. *Geophys. Res. Lett.*, 11:1161–1164, 1984.

[62] D. R. Scott, D. J. Stevenson, and J. A. Whitehead. Observations of solitary waves in a viscously deformable pipe. *Nature*, 319:759–761, 1986.

[63] D. Serre. Spectral stability of periodic solutions of viscous conservation laws: large wavelength analysis. *Comm. Partial Differential Equations*, 30:259–282, 2005.

[64] G. Simpson, M. Spiegelman, and M. I. Weinstein. Degenerate dispersive equations arising in the study of magma dynamics. *Nonlinearity*, 20(1):21–49, 2007.

[65] G. Simpson and M. I. Weinstein. Asymptotic stability of ascending solitary magma waves. *SIAM J. Math. Anal.*, 40(4):1337–1391, 2008.

[66] G. Simpson, M. I. Weinstein, and P. Rosenau. On a Hamiltonian PDE arising in magma dynamics. *Discrete Contin. Dyn. Syst. Ser. B*, 10(4):903–924, 2008.

[67] M. Stanislavova and A. G. Stefanov. Asymptotic stability for spectrally stable Lugiato-Lefever solitons in periodic waveguides. *J. Math. Phys.*, 59(10):101502, 12, 2018.

[68] J. A. Whitehead and K. R. Helfrich. *Magma waves and diapiric dynamics*. Magma Transport and Storage. J. Wiley and Sons, New York, 1990. Michael P. Ryan, ed.

[69] G. B. Whitham. A general approach to linear and non-linear dispersive waves using a lagrangian. *J. Fluid Mech.*, 22:273–283, 1965.

[70] G. B. Whitham. *Linear and nonlinear waves*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.