Alternative Proof of Lemma 2.6

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Proposition 2.6.1. A line L with slope m is a zero line if and only if $m^2 = -1$.

Proof. "$\Rightarrow$" Suppose L is a zero line with slope m. By Lemma 2.2, a zero line has the same slope as the line perpendicular to it, i.e.

$$m = -m^{-1} \Rightarrow m^2 = -1.$$

This may also be shown directly as follows:

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two distinct points on L. Then the slope of L is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

(Note $x_2 \neq x_1$ by Remark 2.1.) Since $\|P_2 - P_1\| = 0$, observe

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 0 \Rightarrow (y_2 - y_1)^2 = -(x_2 - x_1)^2 \Rightarrow m^2 = -1.$$

Note that a slight modification of this direct method can give an alternative proof of Lemma 2.2.

"$\Leftarrow$" Suppose a line L has a slope m such that $m^2 = -1$. This means that L cannot be vertical (as vertical lines do not have a defined slope), which in turn implies that the x-values of the points on L must be distinct. Fix $P_1 = (x_1, y_1) \in L$, fix $P_2 = (x_2, y_2) \in L$ such that $P_2 \neq P_1$, and let $P = (x, y)$ be an arbitrary point in L distinct from P_1. Then,

$$m^2 = -1 \Rightarrow \frac{(y_1 - y)^2}{(x_1 - x)^2} = -1 \Rightarrow (y_1 - y)^2 = -(x_1 - x)^2 \Rightarrow \|P_1 - P\| = 0.$$

We may similarly demonstrate that $\|P_2 - P\| = 0$ for an arbitrary point $P \neq P_2$ in L. In particular, $\|P_2 - P\| = 0 = \|P_1 - P\|$ for all $P \in L$. Hence

$$L \subseteq \mathit{bisector}(P_1, P_2).$$

Since P was arbitrary, we immediately have that $\|P_2 - P_1\| = 0$, which implies $\mathit{bisector}(P_1, P_2)$ is a zero line by definition 1.12. Moreover, this establishes that $P_1, P_2 \in \mathit{bisector}(P_1, P_2)$, which implies L and $\mathit{bisector}(P_1, P_2)$ have two points in common, so we must have

$$L = \mathit{bisector}(P_1, P_2).$$

Therefore, L is a zero line as claimed. $\square$

Remark 2.6.2. Proposition 2.6.1 establishes that zero lines can only occur when $-1 = p - 1$ has a square root.

Lemma 2.6.3. If an arbitrary point P_0 lies on a zero line, there are exactly two zero lines passing through P_0.

Proof. Assume $P_0 = (x_0, y_0) \in \mathbb{F}_q^2$ lies on a zero line L_1. By Proposition 2.6.1, L_1 has a slope m_1 such that $m_1^2 = -1$, which implies -1 has a square root. By Lemma 2.5, we have that -1 has exactly two square roots, and, in particular, there exists a number $m_2 \neq m_1$ such that $m_2^2 = -1$. If we consider the line L_2 with slope m_2 passing through P_0, then Proposition 2.6.1 implies that L_2 is a zero line, which establishes there are at least two zero lines passing through P_0. If there were a third zero line L_3 passing through P_0, then Proposition 2.6.1 implies it would have to have a slope $m_3 \neq m_1, m_2$ such that $m_3^2 = -1$, but this is impossible by Lemma 2.5. Therefore, there are exactly two zero lines passing through the point P_0. $\square$