Problems from the Mathematics 22 Spring 1998 Final Exam

1. Evaluate the following.

(a) $\int \frac{\sin(1-\sqrt{x})}{\sqrt{x}}\,dx$ (b) $\int_0^1 xe^x\,dx$ (c) $\int \frac{1}{x^3+x}\,dx$ (d) $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

2. Determine which of the following improper integrals converge and which diverge. For those which converge, evaluate the integral.

(a) $\int_0^1 \ln x\,dx$ (b) $\int_0^\infty e^{x^2}\,dx$

3. (a) Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

(b) Set up, but **DO NOT EVALUATE**, an integral in terms of x which gives the area of the surface of revolution obtained by rotating the curve in (a) about the y-axis.

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1/\sqrt{1+x^2}$, $x = 0$, $x = 1$, and $y = 0$ about the y-axis.

5. Find the centroid of the region bounded by the curves $y = x^2$, $y = 0$, and $x = 2$.

6. Find all horizontal tangents and vertical tangents for the curve

$$x = 2t^3 + 3t^2, \quad y = t^3 - 3t$$

DO NOT SKETCH THE CURVE.

7. Find the area of the region which lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3/2$.

8. Determine if the given series is conditionally covergent, absolutely convergent, or divergent. Remember to justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{\sin^3(n)}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ (d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$

9. Write down the Maclaurin series (i.e. the Taylor series centered at 0) for each of the following functions. **USE SIGMA NOTATION** (i.e. find a general formula for the terms of the series). You do **NOT** have to show any work for (a), (b), (c), or (d). For (e) and (f), you must show your work, but you may use your answers to (a)–(d).

(a) $\frac{1}{1-x}$ (b) e^x (c) $\sin x$ (d) $\cos x$ (e) $\frac{x}{(1+x)^3}$ (f) $\int_0^x \sin(t^2)\,dt$

10. Find the radius of convergence and the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 2^n}$$

11. Evaluate $\sin(0.1)$ correct to within an error of at most 0.0001.

12. A function $f(x)$ and its derivatives of all orders are defined for all x and $f(0)=1$, $f'(0)=1/\sqrt{2}$, $f''(0)=1/\sqrt{3}$, $\ldots$, $f^{(n)}(0)=1/\sqrt{n+1}$, $\ldots$.

(a) Find the Maclaurin series (i.e. the Taylor series centered at 0) for $f(x)$. **USE SIGMA NOTATION** (i.e. find a general formula for the terms of the series).

(b) Find the radius of convergence and the interval of convergence for the Maclaurin series found in (a).

(c) Assuming $|f^{(n)}(x)| \leq e^{|x|}$ for all x and for all n, show that the Maclaurin series converges to $f(x)$ for all x in the interval of convergence.