Deriving the Friedmann equations from general relativity

The FRW metric in Cartesian coordinates is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + g_{ij}dx^i dx^j = -dt^2 + a(t)^2\left(dx_i^2 + K\frac{x_i^2 dx_i^2}{1-Kx_i^2}\right), \tag{1}$$

where Greek letters run over $\mu, \nu, \ldots = 0, 1, 2, 3$ and latin letters $i, j, \ldots = 1, 2, 3$. The Christoffel symbol $\Gamma^\rho_{\mu\nu}$ is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left[\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right]. \tag{2}$$

For the metric (1) we find the following non-zero components

$$\Gamma^0_{ij} = \frac{\dot{a}(t)}{a(t)}g_{ij}, \tag{3}$$

$$\Gamma^i_{0j} = \frac{\dot{a}(t)}{a(t)}\delta^i_j, \tag{4}$$

$$\Gamma^i_{kj} = \frac{Kx^i g_{kl}}{a(t)^2}. \tag{5}$$

From these we can calculate the Riemann curvature tensor

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}. \tag{6}$$

I will not list all non-zero components here since this is not overly illuminating and we are only interested in the Ricci curvature tensor and the Ricci scalar

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{7}$$

The components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}(t)}{a(t)}, \tag{8}$$

$$R_{0i} = 0, \tag{9}$$

$$R_{ij} = \frac{\ddot{a}(t)a(t) + 2\dot{a}(t)^2 + 2K}{a(t)^2}g_{ij}, \tag{10}$$

where as expected the isotropy and homogeneity of our metric leads to the vanishing of the vector $R_{i0} = 0$ and forces the spacial part to be proportional to the metric $R_{ij} \propto g_{ij}$. The Ricci scalar is given by

$$R = \frac{6\left(a(t)\ddot{a}(t) + \dot{a}(t)^2 + K\right)}{a(t)^2}. \tag{11}$$

We recall from lecture 1 that the energy momentum tensor $T_{\mu\nu}$ is similarly constraint as the Ricci scalar. It can only contain two independent functions of t and its components are

$$\begin{aligned} T_{00} &= \rho(t)\,, && (12)\\ T_{0i} &= 0\,, && (13)\\ T_{ij} &= p(t) g_{ij}\,. && (14) \end{aligned}$$

Now we can solve Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,. \tag{15}$$

First let us look at the (00) component

$$\begin{aligned} -3\frac{\ddot{a}(t)}{a(t)} + \frac{3\left(a(t)\ddot{a}(t) + \dot{a}(t)^2 + K\right)}{a(t)^2} - \Lambda &= 8\pi G\,\rho(t) \\ \frac{3\left(\dot{a}(t)^2 + K\right)}{a(t)^2} - \Lambda &= 8\pi G\,\rho(t)\,. && (16) \end{aligned}$$

Dividing both sides by 3 leads to the first Friedmann equations as given in equation (16) in the Lecture 1 notes

$$\frac{\dot{a}(t)^2 + K}{a(t)^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\,\rho(t)\,. \tag{17}$$

The mixed components ($0i$) all vanish and the pure spacial part takes the form

$$\begin{aligned} \frac{\ddot{a}(t)a(t) + 2\dot{a}(t)^2 + 2K}{a(t)^2} g_{ij} - \frac{3\left(a(t)\ddot{a}(t) + \dot{a}(t)^2 + K\right)}{a(t)^2} g_{ij} + \Lambda g_{ij} &= 8\pi G\, p(t) g_{ij} \\ \left(-2\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}(t)^2 + K}{a(t)^2} + \Lambda\right) g_{ij} &= 8\pi G\, p(t) g_{ij}\,. && (18) \end{aligned}$$

Since the metric $g_{ij} \neq 0$ we can drop it and plug in (17) to get

$$\begin{aligned} -2\frac{\ddot{a}(t)}{a(t)} - \frac{8\pi G}{3}\,\rho(t) - \frac{\Lambda}{3} + \Lambda &= 8\pi G\, p(t) && (19)\\ -2\frac{\ddot{a}(t)}{a(t)} + \frac{2}{3}\Lambda &= 8\pi G\, p(t) + \frac{8\pi G}{3}\,\rho(t)\,. && (20) \end{aligned}$$

Dividing by -2 leads to equation (17) in the lecture notes and the second Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} - \frac{1}{3}\Lambda = -\frac{4\pi G}{3}\left(\rho(t) + 3p(t)\right)\,. \tag{21}$$