Topological aspects in Information-Theoretic Belief Space Planning
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Belief Space Planning (BSP)

- BSP determines optimal non-myopic control action \( U_{t+1} = \arg \min_{U} J(U) \) over the prediction horizon \( t \) at planning time \( k \) with respect to a given objective function \( J \) related to the design task
  \[
  J(U) = \frac{1}{2} \sum_{s} \mathbb{E}[h(X_{s+1}|U_k)] + t_{2} \mathbb{E}[h(X_{t+1})]
  \]
  \( h(X_{t+1}) \) future posterior belief at time \( t + 1 \) based on observations \( Z_{t+1} \) until that time
  \( U_{t+1} \) control applied at time \( t + 1 \)
- Instantiation of a Partially-Observable Markov Decision Process (POMDP)
- Finding optimal solution to POMDP in the most general form is computationally intractable
- In information-theoretic BSP, \( J \) is a function of state uncertainty

Key insight: Any topological representation \( T \) and derived metric which preserves action ordering (the best action) can be used to solve BSP. Exact value of the objective function is not necessary.

Topological BSP (t-BSP)

- We introduce a novel concept, topological belief space planning (t-BSP), that uses topological properties of the underlying factor graph representation of future posterior beliefs to direct a search for an optimal BSP solution
- Topological space is often less dimensional then the embedded state space
- We look for topological representation of the belief and a metric that is highly correlated to \( J \) but much easier to calculate
- No explicit inference required in optimization nor partial state covariance recovery
- Enables planning in high dimensional state spaces

Information-theoretic objective

Minimizing Shannon joint entropy of the posterior Gaussian belief

\[
J(U) = \frac{N}{2} \ln(2\pi\epsilon) + \frac{1}{2} \ln(\mathbb{E}[X_{t+1}])
\]

Proposed topological metrics

- Von Neumann graph entropy and its approximation by a function of node degrees \( d \) (see [1]) faster to calculate, effectively \( O(1) \), worst \( O(n) \)

\[
\Sigma_{VN}(G) = \sum_{i=1}^{n} \frac{d_i}{2} \ln \left( \frac{d_i}{2} \right)
\]

- Function of the number of spanning trees \( \tau(G) \) of a graph motivated by [2] more accurate, computational complexity depends on the graph sparsity and the number of states

\[
\Sigma_{TBSP}(G) = \frac{3}{2} \ln(\tau(G)) - n/2 \ln(N) - \ln(2\pi\epsilon)^2
\]

\( t-BSP \) error \( e(f, s) \) can be calculated from topological metric \( \Sigma_{TBSP} \) and prior maximum likelihood estimate [3]

\[
e(f, s) \leq \Sigma_{max} \text{ where } \Sigma_{max} = \tau(G)\left(\hat{U}(U)\right) - \min_{\alpha} CBF(%) \text{ and } \left\{\begin{array}{l}
\hat{U}(U) = -\Sigma_{TBSP}(U) \text{ and } CBF(%) = -\Sigma_{TBSP}(U) + \frac{1}{2}\tau(U) - \sum_{j=2}^{n} d_j(t_j) - \Psi(U) \end{array}\right\}
\]

Conclusion

topological properties of factor graphs dominantly determine estimation accuracy and enable efficient information-theoretic BSP decision making under some conditions (e.g. linear observation models, large diversity among candidate actions, certain noise properties)
action consistent in other cases, t-BSP enables eliminating sub-optimal actions

References