Some Lemmas in Metric Geometry

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Abstract

This short note summarizes the geometry lemmas appearing in [2].[1]

Some Results Involving Set Distances

For the purpose of this note and for simplicity, we consider only closed subsets of metric spaces in the following lemmas, although all these results can potentially be generalized for subsets that are open or/and closed in the metric space.

Definition 1. Given closed subsets, A, B, of a metric space, (Ψ, d), we define

1. *Separation between the sets:*
$$\mathrm{sep}(A, B) \;=\; \min_{\substack{a\in A,\\ b\in B}} d(a, b)$$

2. *Hausdorff distance between the sets:*
$$d_H(A, B) \;=\; \max\left(\max_{a\in A} \min_{b\in B} d(a, b) \;,\; \max_{b\in B} \min_{a\in A} d(a, b) \right)$$

3. *Diameter of a set:*
$$\mathrm{diam}(A) \;=\; \max_{\substack{a\in A,\\ a'\in A}} d(a, a')$$

Lemma 1. *If (Ψ, d) is a metric space, then for any closed subsets, $P, Q, R \subseteq \Psi$,*
$$\mathrm{sep}(P, Q) \;\leq\; \mathrm{sep}(P, R) \;+\; \mathrm{sep}(R, Q) \;+\; \mathrm{diam}(R) \tag{1}$$

Proof. Let $(p^*, r_1) \in \mathrm{argmin}_{\substack{p\in P,\\ r\in R}} d(p, r)$ (that is, $p^* \in P$, $r_1 \in R$ are a pair of points such that $d(p^*, r_1) = \min_{\substack{p\in P,\\ r\in R}} d(p, r) = \mathrm{sep}(P, R)$). Likewise, let $(q^*, r_2) \in \mathrm{argmin}_{\substack{q\in Q,\\ r\in R}} d(q, r)$ (that is, $d(q^*, r_2) = \mathrm{sep}(R, Q)$). Then,
$$\mathrm{sep}(P, Q) \;\leq\; d(p^*, q^*) \quad \text{(since } \mathrm{sep}(P, Q) = \min_{\substack{p\in P,\\ q\in Q}} d(p, q)\text{)}$$

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[1] To cite the results appearing in this note, please cite [2].

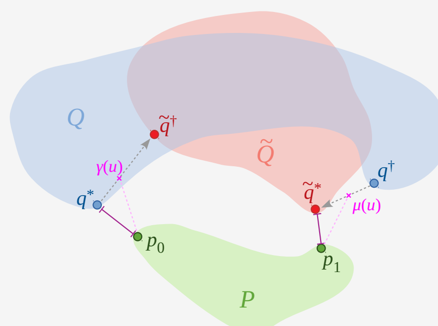
$$
\begin{aligned}
&\leq\ d(p^*, r_1)\ +\ d(r_1, q^*) \quad \text{(triangle inequality.)}\\
&=\ \text{sep}(P, R)\ +\ d(r_1, q^*)\\
&\leq\ \text{sep}(P, R)\ +\ d(r_1, r_2)\ +\ d(q^*, r_2) \quad \text{(triangle inequality.)}\\
&=\ \text{sep}(P, R)\ +\ \text{sep}(R, Q)\ +\ d(r_1, r_2)\\
&\leq\ \text{sep}(P, R)\ +\ \text{sep}(R, Q)\ +\ \text{diam}(R)
\end{aligned} \tag{2}
$$

□

Lemma 2. *If (Ψ, d) is a connected path metric space, then for any closed subsets, $P, Q, \widetilde{Q} \subseteq \Psi$,*

$$
\text{sep}(P, Q)\ \leq\ \text{sep}(P, \widetilde{Q})\ +\ d_H(\widetilde{Q}, Q) \tag{3}
$$

Proof.



Let $(p_0, q^*) \in \text{argmin}_{\substack{p\in P,\\ q\in Q}} d(p, q)$ (that is, $p_0 \in P$, $q^* \in Q$ are a pair of points such that $d(p_0, q^*) = \min_{\substack{p\in P,\\ q\in Q}} d(p, q)$).
Likewise, let $(p_1, \widetilde{q}^*) \in \text{argmin}_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p, q')$.
Furthermore, let $\widetilde{q}^\dagger \in \text{argmin}_{q'\in\widetilde{Q}} d(q^*, q')$ and $q^\dagger \in \text{argmin}_{q\in Q} d(q, \widetilde{q}^*)$.

Consider a shortest path, $\gamma : [0, 1] \to \Psi$, connecting q^* and $\widetilde{q}^\dagger$, and parameterized by the normalized distance from q^*, so that $\gamma(0) = q^*$, $\gamma(1) = \widetilde{q}^\dagger$ and

$$
d(q^*, \gamma(u))\ =\ u\, d(q^*, \widetilde{q}^\dagger) \tag{4}
$$

Likewise, $\mu : [0, 1] \to \Psi$ be the shortest path connecting $q^\dagger$ and $\widetilde{q}^*$, nd parameterized by the normalized distance from $q^\dagger$, so that $\mu(0) = q^\dagger$, $\mu(1) = \widetilde{q}^*$ and $d(q^\dagger, \mu(u)) = u\, d(q^\dagger, \widetilde{q}^*)$. Consequently, since $\mu(u)$ is a point on the shortest path connecting $q^\dagger$ and $\widetilde{q}^*$, we have

$$
d(\mu(u), \widetilde{q}^*)\ =\ d(q^\dagger, \widetilde{q}^*) - d(q^\dagger, \mu(u))\ =\ (1 - u)\, d(q^\dagger, \widetilde{q}^*) \tag{5}
$$

Define $f : [0, 1] \to \mathbb{R}$ as $f(t) = d(p_0, \gamma(t))$, and $g : [0, 1] \to \mathbb{R}$ as $g(t) = d(p_1, \mu(t))$. It's easy to note that both f and g are continuous.

As a consequence, we have the following

$$
f(0) = d(p_0, q^*) = \min_{\substack{p\in P,\\ q\in Q}} d(p, q)\ \leq\ d(p_1, q^\dagger) = g(0)
$$

$$
g(1) = d(p_1, \widetilde{q}^*) = \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p, q')\ \leq\ d(p_0, \widetilde{q}^\dagger) = f(1)
$$

Thus, by intermediate value theorem, there exists a $u \in [0, 1]$ such that $f(u) = g(u)$. That is,

$$
d(p_0, \gamma(u))\ =\ d(p_1, \mu(u)), \qquad \text{for some } u \in [0, 1]. \tag{6}
$$

Using this we have,

$$
\begin{aligned}
\min_{\substack{p\in P,\\ q\in Q}} d(p,q) \;&=\; d(p_0,q^*) \\
&\le\; d(p_0,\gamma(u)) + d(q^*,\gamma(u)) \qquad \text{(triangle inequality.)} \\
&=\; d(p_1,\mu(u)) \;+\; d(q^*,\gamma(u)) \qquad \text{(using (6).)} \\
&\le\; d(p_1,\widetilde{q}^*) + d(\mu(u),\widetilde{q}^*) \;+\; d(q^*,\gamma(u)) \qquad \text{(triangle inequality.)} \\
&=\; \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p,q') \;+\; d(\mu(u),\widetilde{q}^*) + d(q^*,\gamma(u)) \\
&=\; \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p,q') \;+\; (1-u)\, d(q^\dagger,\widetilde{q}^*) + u\, d(q^*,\widetilde{q}^\dagger) \qquad \text{(using (4) and (5).)} \\
&\le\; \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p,q') \;+\; \max\Big(d(q^\dagger,\widetilde{q}^*)\,,\, d(q^*,\widetilde{q}^\dagger)\Big) \\
&=\; \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p,q') \;+\; \max\left(\min_{q\in Q} d(q,\widetilde{q}^*)\,,\, \min_{q'\in\widetilde{Q}} d(q^*,q')\right) \qquad \text{(definitions of } q^\dagger \text{ and } \widetilde{q}^\dagger.\text{)} \\
&\le\; \min_{\substack{p\in P,\\ q'\in \widetilde{Q}}} d(p,q') \;+\; \max\left(\max_{q'\in\widetilde{Q}} \min_{q\in Q} d(q,q')\,,\, \max_{q\in Q} \min_{q'\in\widetilde{Q}} d(q,q')\right) \\
&=\; \mathrm{sep}(P,\widetilde{Q}) \;+\; d_H(\widetilde{Q},Q)
\end{aligned}
$$

□

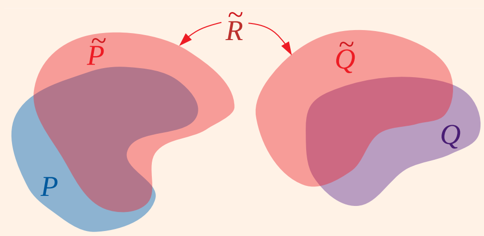
Lemma 3. *Suppose $P, Q, \widetilde{R}$ are closed subsets of a metric space, (Ψ, d), such that*

$$\max_{r'\in\widetilde{R}} \min_{s\in P\cup Q} d(s,r') + d_H(P\cup Q,\widetilde{R}) \;<\; \mathrm{sep}(P,Q) \quad (7)$$

Define, $\widetilde{P},\widetilde{Q}\subseteq\widetilde{R}$, such that

$$\widetilde{P} = \{r'\in\widetilde{R} \mid \min_{s\in P\cup Q} d(s,r') = \min_{p\in P} d(p,r')\}, \quad \textit{and,}$$

$$\widetilde{Q} = \{r'\in\widetilde{R} \mid \min_{s\in P\cup Q} d(s,r') = \min_{q\in Q} d(q,r')\} \qquad (8)$$



Then

1. *$\{\widetilde{P},\widetilde{Q}\}$ constitutes a partition of $\widetilde{R}$,*

2. $\mathrm{argmin}_{s\in P\cup Q}\, d(s,p') \subseteq P,\ \forall p'\in\widetilde{P}$, *and,* $\mathrm{argmin}_{s\in P\cup Q}\, d(s,q') \subseteq Q,\ \forall q'\in\widetilde{Q}$.
(consequently, $\min_{s\in P\cup Q} d(s,p') = \min_{s\in P} d(s,p'),\ \forall p'\in\widetilde{P}$, and, $\min_{s\in P\cup Q} d(s,q') = \min_{s\in Q} d(s,q'),\ \forall q'\in\widetilde{Q}$.)

3. $\mathrm{argmin}_{r'\in\widetilde{R}}\, d(p,r') \subseteq \widetilde{P},\ \forall p\in P$, *and,* $\mathrm{argmin}_{r'\in\widetilde{R}}\, d(q,r') \subseteq \widetilde{Q},\ \forall q\in Q$.
(consequently, $\min_{r'\in\widetilde{R}} d(p,r') = \min_{r'\in\widetilde{P}} d(p,r'),\ \forall p\in P$, and, $\min_{r'\in\widetilde{R}} d(q,r') = \min_{r'\in\widetilde{Q}} d(q,r'),\ \forall q\in Q$.)

4. $d_H(P,\widetilde{P}) \le d_H(P\cup Q,\widetilde{R}),\ \ d_H(Q,\widetilde{Q}) \le d_H(P\cup Q,\widetilde{R})$,

and, $\max\left(d_H(P,\widetilde{P}), d_H(Q,\widetilde{Q})\right) = d_H(P\cup Q,\widetilde{R})$.

5. *If* (Ψ, d) *is a connected path metric space, then* $\mathrm{sep}(\widetilde{P},\widetilde{Q}) \geq \mathrm{sep}(P,Q) - 2\, d_H(P\cup Q,\widetilde{R})$

If the above holds, we say "$\widetilde{R}$ *is a separation-preserving perturbation of* P *and* Q*", and call* $\{\widetilde{P},\widetilde{Q}\}$ *to be the "separation-preserving partition of* $\widetilde{R}$*".*

Proof.

1. We first prove that $\{\widetilde{P},\widetilde{Q}\}$ constitutes of a partition of $\widetilde{R}$.

   Proof for $\widetilde{P}\cup\widetilde{Q} = \widetilde{R}$*:* For a fixed $r'\in\widetilde{R}$, an element of $\mathrm{argmin}_{s\in P\cup Q}\, d(s,r')$ is either in P or in Q. In the former case the point r' will belong to $\widetilde{P}$, while in the later case it will belong to $\widetilde{Q}$ (with the possibility that it belongs to both) due to the definition (8). Thus there does not exist a point $r'\in\widetilde{R}$ that does not belong to either $\widetilde{P}$ or $\widetilde{Q}$.
   Proof for $\widetilde{P}\cap\widetilde{Q} = \emptyset$*:* We prove this by contradiction. If possible, let $\rho'\in\widetilde{P}\cap\widetilde{Q}$. Since $\rho'\in\widetilde{P}$, due to definition (8), there exists a $p_1\in P$ such that $\min_{s\in P\cup Q} d(s,\rho') = d(p_1,\rho')$. Likewise, there exists a $q_1\in Q$ such that $\min_{s\in P\cup Q} d(s,\rho') = d(q_1,\rho')$. Thus,

$$\begin{aligned}
2\min_{s\in P\cup Q} d(s,\rho') &= d(p_1,\rho') + d(q_1,\rho') \\
&\geq d(p_1,q_1) \qquad \text{(tringle inequality.)} \\
&\geq \min_{\substack{p\in P,\\ q\in Q}} d(p,q) \qquad \text{(since } p_1\in P, q_1\in Q.\text{)} \\
\Rightarrow\ 2\max_{r'\in\widetilde{R}}\min_{s\in P\cup Q} d(s,r') &\geq \min_{\substack{p\in P,\\ q\in Q}} d(p,q) \\
\Rightarrow\ \max_{r'\in\widetilde{R}}\min_{s\in P\cup Q} d(s,r') + d_H(P\cup Q,\widetilde{R}) &\geq \mathrm{sep}(P,Q)
\end{aligned}$$

   This contradicts the assumption (7) of the Lemma. Hence there cannot exist a $\rho'\in\widetilde{P}\cap\widetilde{Q}$. Thus $\widetilde{P}\cap\widetilde{Q} = \emptyset$.

2. We next prove $\mathrm{argmin}_{s\in P\cup Q}\, d(s,p') \subseteq P,\ \forall p'\in\widetilde{P}$. We do this by contradiction.

   If possible, suppose there exists a $p'\in\widetilde{P}$ such that $\mathrm{argmin}_{s\in P\cup Q}\, d(s,p') \not\subseteq P$. Then there exists a $q\in Q$ such that $\min_{s\in P\cup Q} d(s,p') = d(q,p')$. But $d(q,p') \geq \min_{s\in Q} d(s,p') \geq \min_{s\in P\cup Q} d(s,p')$. This implies $\min_{s\in P\cup Q} d(s,p') = \min_{s\in Q} d(s,p')$. Due to definition of $\widetilde{Q}$ in (8) this implies $p'\in\widetilde{Q}$. However, we have already shown that $\widetilde{P}\cap\widetilde{Q} = \emptyset$. This leads to a contradiction. Thus $\mathrm{argmin}_{s\in P\cup Q}\, d(s,p') \subseteq P,\ \forall p'\in\widetilde{P}$.

   Likewise we can prove $\mathrm{argmin}_{s\in P\cup Q}\, d(s,q') \subseteq Q,\ \forall q'\in\widetilde{Q}$.

3. We next prove $\mathrm{argmin}_{r'\in\widetilde{R}}\, d(p,r') \subseteq \widetilde{P},\ \forall p\in P$. We do this by contradiction.

   If possible, suppose there exists a $p_3\in P$ such that $\mathrm{argmin}_{r'\in\widetilde{R}}\, d(p_3,r') \not\subseteq \widetilde{P}$. Then there exists a $\rho'\in\widetilde{Q}$ such that $\min_{r'\in\widetilde{R}} d(p_3,r') = d(p_3,\rho')$.

Again, due to the definition of $\widetilde{Q}$ in (8), for any $\rho' \in \widetilde{Q}$ there exists a $q_3 \in Q$ such that $d(q_3, \rho') = \min_{s \in P \cup Q} d(s, \rho')$.

Thus,

$$\begin{aligned}
\min_{r' \in \widetilde{R}} d(p_3, r') + \min_{s \in P \cup Q} d(s, \rho') &= d(p_3, \rho') + d(q_3, \rho') \\
&\geq d(p_3, q_3) \qquad \text{(tringle inequality.)} \\
&\geq \min_{\substack{p \in P, \\ q \in Q}} d(p, q) \qquad \text{(since } p_3 \in P, q_3 \in Q.\text{)}
\end{aligned}$$

$$\Rightarrow \max_{s \in P \cup Q} \min_{r' \in \widetilde{R}} d(s, r') + \max_{r' \in \widetilde{R}} \min_{s \in P \cup Q} d(s, r') \geq \min_{\substack{p \in P, \\ q \in Q}} d(p, q)$$

$$\Rightarrow d_H(P \cup Q, \widetilde{R}) + \max_{r' \in \widetilde{R}} \min_{s \in P \cup Q} d(s, r') \geq \operatorname{sep}(P, Q)$$

This contradicts the assumption (7) of the Lemma. Hence there cannot exist a $p_3 \in P$ such that $\operatorname{argmin}_{r' \in \widetilde{R}} d(p_3, r') \nsubseteq \widetilde{P}$. Thus $\operatorname{argmin}_{r' \in \widetilde{R}} d(p, r') \subseteq \widetilde{P},\ \forall p \in P$.

Likewise we can prove $\operatorname{argmin}_{r' \in \widetilde{R}} d(q, r') \subseteq \widetilde{Q},\ \forall q \in Q$.

4. Since $\operatorname{argmin}_{s \in P \cup Q} d(s, p') \subseteq P,\ \forall p' \in \widetilde{P}$, we have $\min_{s \in P \cup Q} d(s, p') = \min_{p \in P} d(p, p'),\ \forall p' \in \widetilde{P}$. Thus, $\max_{p' \in \widetilde{P}} \min_{p \in P} d(p, p') = \max_{p' \in \widetilde{P}} \min_{s \in P \cup Q} d(s, p')$.

   Likewise, since $\operatorname{argmin}_{r' \in \widetilde{R}} d(p, r') \subseteq \widetilde{P},\ \forall p \in P$, we have $\max_{p \in P} \min_{p' \in \widetilde{P}} d(p, p') = \max_{p \in P} \min_{r' \in \widetilde{R}} d(p, r')$.

   Thus,

$$\begin{aligned}
d_H(P, \widetilde{P}) &= \max\left(\max_{p \in P} \min_{p' \in \widetilde{P}} d(p, p'),\ \max_{p' \in \widetilde{P}} \min_{p \in P} d(p, p') \right) \\
&= \max\left(\max_{p \in P} \min_{r' \in \widetilde{R}} d(p, r'),\ \max_{p' \in \widetilde{P}} \min_{s \in P \cup Q} d(s, p') \right) \qquad (9) \\
&\leq \max\left(\max_{s \in P \cup Q} \min_{r' \in \widetilde{R}} d(s, r'),\ \max_{r' \in \widetilde{R}} \min_{s \in P \cup Q} d(s, r') \right) \qquad \text{(since } P \subseteq P \cup Q,\ \widetilde{P} \subseteq \widetilde{R}.\text{)} \\
&= d_H(P \cup Q, \widetilde{R})
\end{aligned}$$

   Similarly we can show,

$$\begin{aligned}
d_H(Q, \widetilde{Q}) &= \max\left(\max_{q \in Q} \min_{r' \in \widetilde{R}} d(q, r'),\ \max_{q' \in \widetilde{Q}} \min_{s \in P \cup Q} d(s, q') \right) \qquad (10) \\
&\leq d_H(P \cup Q, \widetilde{R})
\end{aligned}$$

   Again, from (9) and (10),

$$\max\left(d_H(P, \widetilde{P}), d_H(Q, \widetilde{Q}) \right) = \max\left(\max_{p \in P} \min_{r' \in \widetilde{R}} d(p, r'),\ \max_{q \in Q} \min_{r' \in \widetilde{R}} d(q, r'),\right.$$

$$\max_{p'\in\widetilde{P}} \min_{s\in P\cup Q} d(s,p')\,,\ \max_{q'\in\widetilde{Q}} \min_{s\in P\cup Q} d(s,q')\Bigg)$$

$$= \max\left(\max_{p\in P\cup Q} \min_{r'\in\widetilde{R}} d(p,r')\,,\ \max_{p'\in\widetilde{P}\cup\widetilde{Q}} \min_{s\in P\cup Q} d(s,p')\right)$$

$$= d_H(P\cup Q,\widetilde{R}) \qquad \text{(since } \widetilde{P}\cup\widetilde{Q}=\widetilde{R}\text{)}$$

5.

$$\begin{aligned} \mathrm{sep}(\widetilde{P},\widetilde{Q}) \geq\ & \mathrm{sep}(\widetilde{P},Q) - d_H(Q,\widetilde{Q}) \quad \text{(using Lemma 2.)} \\ \geq\ & \mathrm{sep}(P,Q) - d_H(P,\widetilde{P}) - d_H(Q,\widetilde{Q}) \quad \text{(using Lemma 2.)} \\ \geq\ & \mathrm{sep}(P,Q) - 2\,d_H(P\cup Q,\widetilde{R}) \\ & \text{(since } d_H(P,\widetilde{P}) \leq d_H(P\cup Q,\widetilde{R}) \text{ and } d_H(Q,\widetilde{Q}) \leq d_H(P\cup Q,\widetilde{R})\text{.)} \end{aligned}$$

□

Corollary 1. *If $P, Q, \widetilde{R}$ are closed subsets of a metric space, (Ψ, d), such that $d_H(P\cup Q,\widetilde{R}) < \frac{1}{2}\,\mathrm{sep}(P,Q)$, then $\widetilde{R}$ is a separation-preserving perturbation of P and Q.*

As a consequence, the separation-preserving partition, $\{\widetilde{P},\widetilde{Q}\}$, of $\widetilde{R}$ as defined in (8) satisfies properties '1' to '4' in Lemma 3, as well as property '5' (if (Ψ, d) is a connected path metric space) with an additional inequality:

$$\mathrm{sep}(\widetilde{P},\widetilde{Q}) \ \geq\ \mathrm{sep}(P,Q) - 2\,d_H(P\cup Q,\widetilde{R}) \ >\ 0$$

Proof. The result follows directly from Lemma 3 by observing that

$$\max_{r'\in\widetilde{R}} \min_{s\in P\cup Q} d(s,r') + d_H(P\cup Q,\widetilde{R}) \ \leq\ 2\,d_H(P\cup Q,\widetilde{R}) \ <\ \mathrm{sep}(P,Q)$$

□

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