Relationship Between Gradient of Distance Functions and Tangents to Geodesics

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In the discussions that follow, we will assume summation over repeated indices, i and j, following Einstein summation convention.

Proposition 1. *Let $C = (U, \phi)$ be a coordinate chart on a open subset U of a D-dimensional manifold, Ω with coordinate variables $u^1, u^2, \cdots, u^D$. Suppose U is Riemannian everywhere, equipped with a metric η (which is assume to be non-singular everywhere* - i.e. *$\eta_{\bullet\bullet}$ is positive definite), and the geodesic connecting any two points in U lies entirely in U.*

Let $d : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$ be the distance function in $U \subseteq \Omega$ in terms of the coordinate chart C (i.e. $d(\mathbf{q}, \mathbf{w})$, for $\mathbf{q}, \mathbf{w} \in Img(\phi) \subseteq \mathbb{R}^D$, is the length of the shortest path connecting $\phi^{-1}(\mathbf{q})$ and $\phi^{-1}(\mathbf{w})$ in $U \subseteq \Omega$.). Suppose inside $Img(\phi)$, the distance d is induced by the Riemannian metric η, is smooth everywhere, and suppose there exists an unique geodesic of length $d(\mathbf{q}, \mathbf{w})$ connecting any two points $\mathbf{q}, \mathbf{w} \in Img(\phi)$.

Then the following is true for every $\mathbf{q}, \mathbf{w} \in Img(\phi) \subseteq \mathbb{R}^D$ and every coordinate chart, C, defined on U (Figure 1(a)),

$$\left[\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}, \mathbf{u}) \bigg|_{\mathbf{u}=\mathbf{w}} \right]_i \equiv \frac{\partial}{\partial u^i} d(\mathbf{q}, \mathbf{u}) \bigg|_{\mathbf{u}=\mathbf{w}} = \frac{\eta_{ij}(\mathbf{w}) \, z^j_{\mathbf{qw}}}{\sqrt{\eta_{mn}(\mathbf{w}) \, z^m_{\mathbf{qw}} z^n_{\mathbf{qw}}}}$$

where, $\mathbf{z}_{\mathbf{qw}} = [z^1_{\mathbf{qw}}, z^2_{\mathbf{qw}}, \cdots, z^D_{\mathbf{qw}}]^T$ is a normalized coefficient vector of the tangent vector at $\mathbf{w}$ to the shortest geodesic connecting $\mathbf{q}$ to $\mathbf{w}$, and by $\left[\frac{\partial f}{\partial \mathbf{u}}\right]_i$ we mean the i^{th} component of $\left[\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, \cdots, \frac{\partial f}{\partial u^D}\right]$.

Proof. For notational convenience, let us define $g(\mathbf{u}) := d(\mathbf{q}, \mathbf{u}), \ \forall \mathbf{u} \in Img(\phi)$.

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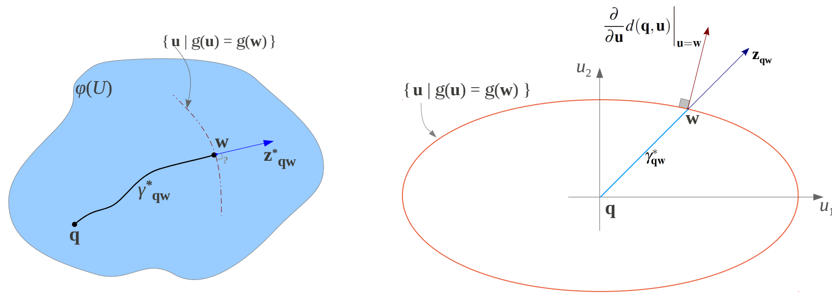


(a) Illustration for Proposition 1. It establishes a relation between the tangent to the geodesic $\gamma^*_{\mathbf{qw}}$ at $\mathbf{w}$, and the normal to the surface $\{\mathbf{u}|g(\mathbf{u}) = g(\mathbf{w})\}$ at $\mathbf{w}$.

(b) Illustration in a simple non-Euclidean, anisotropic metric. Note that the normal to the ellipse is not parallel to the tangent to the geodesic, $\mathbf{z}_{\mathbf{wq}}$. It is however parallel to the cotangent, $\mathbf{z}^*_{\mathbf{wq}}$, with coefficients $z^*_{j,\mathbf{qw}} = \eta_{ij}(\mathbf{w})\, z^j_{\mathbf{qw}}$.

Fig. 1 Relationship between tangent to a geodesic and the derivative of the distance function.

Consider g as a function from $\mathbb{R}^D$ to $\mathbb{R}^+$ with an unique minima at $\mathbf{q}$. Let $\gamma_{\mathbf{qw}}$ represents any arbitrary curve in $Img(\phi) \subseteq \mathbb{R}^D$ connecting $\mathbf{q}$ to $\mathbf{w}$. By the *fundamental theorem of calculus* and using the fact that $g(\mathbf{q}) = 0$, we have,

$$I(\gamma_{\mathbf{qw}}) \;:=\; g(\mathbf{w}) \;=\; \int_{\gamma_{\mathbf{qw}}} \frac{\partial}{\partial \mathbf{u}} g(\mathbf{u}) \cdot \mathrm{d}\mathbf{u} \;\equiv\; \int_{\gamma_{\mathbf{qw}}} \frac{\partial g(\mathbf{u})}{\partial u^i}\, \mathrm{d}u^i \tag{1}$$

where, $[\,\mathrm{d}u^1,\, \mathrm{d}u^2, \cdots,\, \mathrm{d}u^D]$ is the coefficient vector (in chart C) of an infinitesimal element along the tangent to the curve.

Now, the length of the curve $\gamma_{\mathbf{qw}}$ is given by

$$L(\gamma_{\mathbf{qw}}) \;:=\; \int_{\gamma_{\mathbf{qw}}} \sqrt{\eta_{ij}(\mathbf{u})\;\, \mathrm{d}u^i\, \mathrm{d}u^j} \tag{2}$$

By definition, the value of $L(\gamma_{\mathbf{qw}})$ is minimum when $\gamma_{\mathbf{qw}}$ is the shortest geodesic (which is unique by hypothesis – call it $\gamma^*_{\mathbf{qw}}$) connecting $\mathbf{q}$ and $\mathbf{w}$, and the minimum value is clearly $g(\mathbf{w})$ (by definition of g). Thus,

$$\begin{array}{c} L(\gamma_{\mathbf{qw}}) \;\geq\; I(\gamma_{\mathbf{qw}}) \;\; [= g(\mathbf{w}),\text{ a const. independent of } \gamma_{\mathbf{qw}}]\;, \\ \text{equality holds when } \gamma_{\mathbf{qw}} = \gamma^*_{\mathbf{qw}} \end{array} \tag{3}$$

Now, consider a family of infinitesimal elements of $\mathbb{R}^D$ represented by the coefficient vector $\mathrm{d}\mathbf{u} = [\,\mathrm{d}u^1,\, \mathrm{d}u^2, \cdots,\, \mathrm{d}u^D]^T$ located at an arbitrary point $\mathbf{u} \in Img(\phi)$ such that $\mathbf{u} + \mathrm{d}\mathbf{u}$ lies inside $Img(\phi)$. From the triangle inequality of d (since it is induced by a Riemannian metric) we have,

$$\begin{aligned}
d(\mathbf{q},\mathbf{u}+\mathrm{d}\mathbf{u}) &\leq d(\mathbf{q},\mathbf{u})+d(\mathbf{u},\mathbf{u}+\mathrm{d}\mathbf{u}) \\
\implies d(\mathbf{q},\mathbf{u}+\mathrm{d}\mathbf{u})-d(\mathbf{q},\mathbf{u}) &\leq d(\mathbf{u},\mathbf{u}+\mathrm{d}\mathbf{u}) \\
\implies \left.\frac{\partial}{\partial\mathbf{u}}g(\mathbf{u})\right|_{\mathbf{u}}\cdot\mathrm{d}\mathbf{u} &\leq \sqrt{\eta_{ij}(\mathbf{u})\ \mathrm{d}u^i\,\mathrm{d}u^j} \qquad (4)\\
\implies \frac{\partial g(\mathbf{u})}{\partial u_i}\,\mathrm{d}u^i &\leq \sqrt{\eta_{ij}(\mathbf{u})\ \mathrm{d}u^i\,\mathrm{d}u^j} = \frac{\eta_{ij}(\mathbf{u})\,\mathrm{d}u^i\,\mathrm{d}u^j}{\sqrt{\eta_{mn}(\mathbf{u})\ \mathrm{d}u^m\,\mathrm{d}u^n}}
\end{aligned}$$

Equality of the triangle inequality in (4) of course holds when $\mathbf{u}$ lies on the geodesic connecting $\mathbf{q}$ and $\mathbf{u}+\mathrm{d}\mathbf{u}$.

Now consider a curve $\gamma'_{\mathbf{qw}}$ (connecting $\mathbf{q}$ and $\mathbf{w}$) with infinitesimal elements $\mathrm{d}\mathbf{u}$ along the tangents to the curve. If there exists at least one point along that curve on which the inequality in (4) is not an equality, then the integrals of the quantities on the two sides of the inequality will not be equal. That is, for such a curve we will have $I(\gamma'_{\mathbf{qw}}) < L(\gamma'_{\mathbf{qw}})$.

But we know that there does exist a curve, $\gamma^*_{\mathbf{qw}}$, such that $I(\gamma^*_{\mathbf{qw}}) = L(\gamma^*_{\mathbf{qw}})$ does hold. Thus, for that curve it should be true that at each and every point of the curve the equality of (4) holds true. Thus we have essentially shown that for the geodesic, $\gamma^*_{\mathbf{qw}}$, at each and every point of the curve the following holds

$$\frac{\partial g(\mathbf{u})}{\partial u_i}\,\mathrm{d}u^i = \frac{\eta_{ij}(\mathbf{u})\,\mathrm{d}u^i\,\mathrm{d}u^j}{\sqrt{\eta_{mn}(\mathbf{u})\ \mathrm{d}u^m\,\mathrm{d}u^n}}$$

where $\mathrm{d}\mathbf{u}$ are of course infinitesimal elements at $\mathbf{u}$ along the tangent to $\gamma^*_{\mathbf{qw}}$.

One can normalize by dividing by $\|\mathrm{d}\mathbf{u}\|_2$ to obtain

$$\frac{\partial g(\mathbf{u})}{\partial u_i} z^i_{\mathbf{qu}} = \frac{\eta_{ij}(\mathbf{u}) z^i_{\mathbf{qu}} z^j_{\mathbf{qu}}}{\sqrt{\eta_{mn}(\mathbf{u})\ z^m_{\mathbf{qu}} z^n_{\mathbf{qu}}}} \qquad (5)$$

where $z^i_{\mathbf{qu}}$ is the i^{th} component of the tangent vector at $\mathbf{u}$ to the geodesic connecting $\mathbf{q}$ to $\mathbf{u}$, which due to our assumption is unique. We note that the right-hand-side of the above equation represents a scalar field (call it S). Also, $z^i_{\mathbf{qu}}$ (which are functions of $\mathbf{u}$) represent the coefficients of a contravariant vector field in $(U-\mathbf{q})$. Thus, writing X_i for $\frac{\partial g(\mathbf{u})}{\partial u_i}$, one can rewrite Equation (5) as

$$X_i z^i_{\mathbf{qu}} = S(\mathbf{u}) \qquad (6)$$

where we need to solve for the coefficients $X_i(\mathbf{u}) := \frac{\partial g(\mathbf{u})}{\partial u_i}$. Of course a particular solution is

$$X_{0,i}(\mathbf{u}) = \frac{\eta_{ij}(\mathbf{u}) z^j_{\mathbf{qu}}}{\sqrt{\eta_{mn}(\mathbf{u})\ z^m_{\mathbf{qu}} z^n_{\mathbf{qu}}}} \qquad (7)$$

These coefficients clearly transform as coefficients of a covariant vector field. Moreover, the contravariant vector field corresponding to this covariant field (*i.e.* the vectors with coefficients $X^{0,i} = \eta^{ij} X_{0,j}$) is parallel to $z^i_{\mathbf{qu}}$. From this we can infer that the solution mentioned in (7) is the only solution of (6) that transforms as coefficients of a covariant vector field (This is because of the following: Every covariant transformation of X_i corresponds to an unique contravariant transformation of X^i. Again, the general solutions of X^i need to be such that $\eta_{ij} z^i_{\mathbf{qu}} X^j = \eta_{ij} z^i_{\mathbf{qu}} X^{0,j} = \beta \eta_{ij} z^i_{\mathbf{qu}} z^j_{\mathbf{qu}}$ for some scalar field β. This needs to be true in every coordinate chart. Due to positive definiteness of $\eta_{\bullet\bullet}$, this is possible only with X^j parallel to $z^j_{\mathbf{qu}}$, which also fixes the scalar multiple β since we have the known scalar field, S.).

Thus we have

$$\frac{\partial g(\mathbf{u})}{\partial u_i} = \frac{\eta_{ij}(\mathbf{u}) z^j_{\mathbf{qu}}}{\sqrt{\eta_{mn}(\mathbf{u})\, z^m_{\mathbf{qu}} z^n_{\mathbf{qu}}}} \tag{8}$$

Thus, by specializing for $\mathbf{u} = \mathbf{w}$, we obtain the proposed result. $\square$

If we define $g(\mathbf{u}) := d(\mathbf{q}, \mathbf{u})$, $\forall \mathbf{u} \in Img(\phi)$ (*i.e.*, $g(\mathbf{w})$ is the length of the shortest geodesic connecting $\mathbf{q}$ to $\mathbf{w}$), the statement of the proposition essentially implies that the normals to the constant g surfaces in $\mathbb{R}^D$ are parallel to the cotangents to the geodesics. This is illustrated in Figure 1(a). The statement of the proposition essentially expresses the gradient of the distance function d (with respect to one of its arguments) in terms of the tangent to the geodesic connecting two points.

Examples:

1. We note that when the metric is Euclidean in the given chart (*i.e.* $\eta_{ij} = \delta_{ij}$ everywhere as was the case in [2]), the result of the proposition simply reduces to $\frac{\partial}{\partial u^i} d(\mathbf{q}, \mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}} = z^i_{\mathbf{qw}}$. This is no surprise since we know that the vector $\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}, \mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}}$ is essentially an unit normal to the sphere with center $\mathbf{q}$ (which is the surface of constant $d(\mathbf{q}, \mathbf{u})$) at the point $\mathbf{u} = \mathbf{w}$, which is well-known to be parallel to the straight line connecting $\mathbf{q}$ to $\mathbf{w}$ (a radial line of the sphere).
2. If the metric is locally isotropic in the given chart (*i.e.* if the matrix representation of the metric is a multiple of the identity matrix at every point), and can be written as $\eta_{ij}(\mathbf{q}) = \zeta(\mathbf{q}) \delta_{ij}$ for some $\zeta : \mathbb{R}^D \to \mathbb{R}$, then the result of the proposition reduces to $\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}, \mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}} = \sqrt{\zeta(\mathbf{w})}\ \mathbf{z}^T_{\mathbf{qw}}$ (where, $\mathbf{z}^T_{\mathbf{qw}} = [z^1_{\mathbf{qw}}, z^2_{\mathbf{qw}}, \cdots, z^D_{\mathbf{qw}}]$ is the transpose of the coefficient vector, $\mathbf{z}_{\mathbf{qw}}$, of the tangent to the geodesic).
3. Finally, we consider a simple, yet nontrivial, example of a non-Euclidean, anisotropic metric. Consider the metric $\eta_{\bullet\bullet} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Since the Christoffel symbols vanish in this coordinate chart, one can infer from the geodesic equation that the geodesics are essentially represented by straight lines when plotted with u^i as orthogonal axes (Figure 1(b)). However, the curves of constant distance from $\mathbf{q}$ become ellipses centered at $\mathbf{q}$ and with aspect ratio of 2. Now consider the point $\mathbf{w} = \mathbf{q} + [1, 1]^T$. A direct computation of the normal at this point to the ellipse,

$(u^1-q^1)^2/4+(u^2-q^2)^2=c$, passing through this point, reveals the coefficient co-vector of $\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q},\mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}}$ to be parallel to $[\frac{1}{2},2]$. However, the coefficient vector of the tangent to the geodesic is $\mathbf{z_{qw}}=[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]^T$. This gives the following: $\sqrt{\eta_{mn}(\mathbf{w})\, z^m_{\mathbf{qw}} z^n_{\mathbf{qw}}}=\sqrt{\frac{5}{2}}$, $\quad z_{1,\mathbf{qw}}=\sum_j \eta_{1j}\, z^j_{\mathbf{qw}}=\frac{1}{\sqrt{2}}$, $\quad z_{2,\mathbf{qw}}=\sum_j \eta_{2j}\, z^j_{\mathbf{qw}}=2\sqrt{2}$. Thus, the coefficient co-vector of $\mathbf{z}^*_{\mathbf{qw}}$ is parallel to $[\frac{1}{\sqrt{2}},2\sqrt{2}]$. This indeed is parallel to $[\frac{1}{2},2]$. The exact computation of the scalar multiple will require a more careful computation of $\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q},\mathbf{u})$.

Corollary 1. *Let $C=(V,\psi)$ be a coordinate chart on a open subset V of a D-dimensional manifold, Ω. Let $d:\mathbb{R}^D\times\mathbb{R}^D\to\mathbb{R}$ be the distance function on V in terms of the coordinate chart C (i.e. $d(\mathbf{q},\mathbf{w})$, for $\mathbf{q},\mathbf{w}\in Img(\psi)\subseteq\mathbb{R}^D$, is the distance between $\psi^{-1}(\mathbf{q})$ and $\psi^{-1}(\mathbf{w})$ in $V\subseteq\Omega$).*

We are given $\mathbf{q}_0,\mathbf{w}_0\in Img(\psi)\subseteq\mathbb{R}^D$. Suppose there exists a open neighborhood $\mathscr{B}_{\mathbf{w}_0}\subseteq\mathbb{R}^D$ of $\mathbf{w}_0$ (Figure 2(a)) such that,

a. $g(\cdot):=d(\mathbf{q}_0,\cdot)$ is smooth everywhere in $\mathscr{B}_{\mathbf{w}_0}$,
b. The distance function restricted to $\mathscr{B}_{\mathbf{w}_0}\times\mathscr{B}_{\mathbf{w}_0}$ is induced by a Riemannian metric, η (non-singular), such that for any two $\mathbf{u},\mathbf{v}\in\mathscr{B}_{\mathbf{w}_0}$, there is an unique shortest geodesic $\gamma_{\mathbf{uv}}$ of length $d(\mathbf{u},\mathbf{v})$.
c. A shortest path (of length $d(\mathbf{q}_0,\mathbf{w}_0)$) is defined between $\mathbf{q}_0$ and $\mathbf{w}_0$, such that the part of the shortest path connecting $\mathbf{q}_0$ and $\mathbf{w}_0$ that lies inside $\mathscr{B}_{\mathbf{w}_0}$ is unique,

Then the following holds,

$$\left[\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}_0,\mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}_0}\right]_i \equiv \frac{\partial}{\partial u^i} d(\mathbf{q}_0,\mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}_0} = \frac{\eta_{ij}(\mathbf{w}_0)\, z^j_{\mathbf{q}_0\mathbf{w}_0}}{\sqrt{\eta_{mn}(\mathbf{w}_0)\, z^m_{\mathbf{q}_0\mathbf{w}_0} z^n_{\mathbf{q}_0\mathbf{w}_0}}}$$

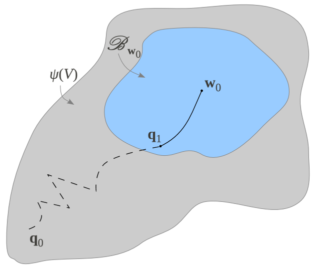
Note that the derivative $\frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}_0,\mathbf{u})\Big|_{\mathbf{u}=\mathbf{w}_0}$ is defined due to assumption 'a.', and the tangent $\mathbf{z}_{\mathbf{q}_0\mathbf{w}_0}$ exists due to assumptions 'b.' and 'c.'.

Proof. Consider a shortest path $\gamma^*_{\mathbf{q}_0\mathbf{w}_0}$ connecting $\mathbf{q}_0$ and $\mathbf{w}_0$. Let $\mathbf{q}_1(\neq\mathbf{w}_0)$ be a point on this path (between $\mathbf{q}_0$ and $\mathbf{w}_0$) that lies inside $\mathscr{B}_{\mathbf{w}_0}$ (which we can always find since $\mathscr{B}_{\mathbf{w}_0}$ is open).

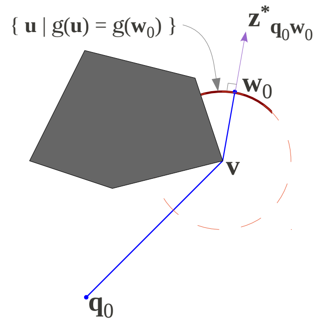
From the very definition of shortest path we have $\gamma^*_{\mathbf{q}_0\mathbf{w}_0}=\gamma^*_{\mathbf{q}_0\mathbf{q}_1}\cup\gamma^*_{\mathbf{q}_1\mathbf{w}_0}$, for some shortest path, $\gamma^*_{\mathbf{q}_0\mathbf{q}_1}$, connecting $\mathbf{q}_0$ and $\mathbf{q}_1$, and the shortest geodesic, $\gamma^*_{\mathbf{q}_1\mathbf{w}_0}$, connecting $\mathbf{q}_1$ and $\mathbf{w}_0$ (which is unique by assumption '*b.*'). Thus it follows that,

$$\mathbf{z}_{\mathbf{q}_0\mathbf{w}_0}=\mathbf{z}_{\mathbf{q}_1\mathbf{w}_0} \tag{9}$$

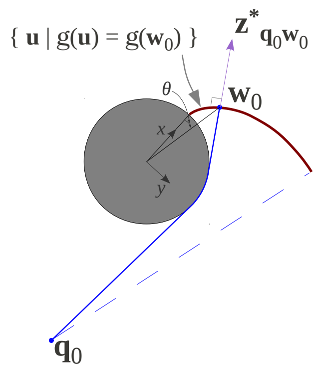
Again, by triangle inequality, for any $\mathbf{u}\in\mathscr{B}_{\mathbf{w}_0}$



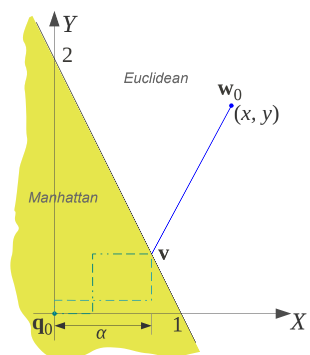
(a) Illustration for Corollary 1. Much of the pathologies outside $\mathscr{B}_{\mathbf{w}_0}$ do not effect the result of Proposition 1 holding for $\mathbf{q}_0$ and $\mathbf{w}_0$.



(b) The simplest example is that of a space that is equipped with Euclidean metric everywhere, but is punctured by a polygonal obstacles. This was the case considered in [2].



(c) A more interesting case is that of involutes generated in locally Euclidean space using the boundaries of arbitrary obstacles as the generating curves.



(d) An example involving Manhattan and Euclidean metrics in two different regions. The distance function between points lying in the two different regions is given by $d(\mathbf{q}_0, \mathbf{w}_0) = \min_{\mathbf{v}} (d_{man}(\mathbf{q}_0, \mathbf{v}) + d_{eu}(\mathbf{v}, \mathbf{w}_0))$, where $\mathbf{v}$ is a point lying on the boundary of the two regions.

Fig. 2 Corollary 1 and examples validating it.

$$\begin{aligned} d(\mathbf{q}_0, \mathbf{u}) &\leq d(\mathbf{q}_0, \mathbf{q}_1) + d(\mathbf{q}_1, \mathbf{u}) \\ \Rightarrow g(\mathbf{u}) &\leq h(\mathbf{u}) \end{aligned} \tag{10}$$

where $h(\mathbf{u}) := d(\mathbf{q}_0, \mathbf{q}_1) + d(\mathbf{q}_1, \mathbf{u})$ and $g(\mathbf{u}) := d(\mathbf{q}_0, \mathbf{u})$.
However, equality does hold when $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{u}$ lie on the same shortest path. This, in particular, is true when $\mathbf{u} = \mathbf{w}_0$ (due to our choice of $\mathbf{q}_1$). Now, by our assumptions, both g and h are smooth at $\mathbf{w}_0$ (assumptions '$a.$' and '$b.$' respectively). Thus we have

$g(\mathbf{u}) \leq h(\mathbf{u})$, and at $\mathbf{u} = \mathbf{w}_0$ they satisfy equality and are smooth. This implies the derivatives of the functions at $\mathbf{w}_0$ should be same,

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}} g(\mathbf{u})\bigg|_{\mathbf{u}=\mathbf{w}_0} &= \frac{\partial}{\partial \mathbf{u}} h(\mathbf{u})\bigg|_{\mathbf{u}=\mathbf{w}_0} \\ \Rightarrow \frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}_0, \mathbf{u})\bigg|_{\mathbf{u}=\mathbf{w}_0} &= \frac{\partial}{\partial \mathbf{u}} d(\mathbf{q}_1, \mathbf{u})\bigg|_{\mathbf{u}=\mathbf{w}_0} \end{aligned} \tag{11}$$

Now, $\mathscr{B}_{\mathbf{w}_0}$ satisfies the conditions for U in Proposition 1, and $\mathbf{q}_1$ and $\mathbf{w}_0$ are points inside it. Thus by Proposition 1,

$$\frac{\partial}{\partial u^i} d(\mathbf{q}_1, \mathbf{u})\bigg|_{\mathbf{u}=\mathbf{w}_0} = \frac{\eta_{ij}(\mathbf{w}_0)\; z^j_{\mathbf{q}_1 \mathbf{w}_0}}{\sqrt{\eta_{mn}(\mathbf{w}_0)\; z^m_{\mathbf{q}_1 \mathbf{w}_0} z^n_{\mathbf{q}_1 \mathbf{w}_0}}} \tag{12}$$

Substituting from (9) and (11) into (12) we obtain the proposed result. $\square$

The statement of this Corollary is a significant generalization of Proposition 1. Here we make assumption of a Riemannian metric only in the neighborhood of $\mathbf{w}_0$ (Figure 2(a)). This will enable us to use the result for locally Riemannian manifolds with pathologies outside local neighborhoods (*e.g.* boundaries/holes/punctures/obstacles – the kind of spaces we are most interested in), as well as opens up possibilities for more general metric spaces that may not be Riemannian outside $\mathscr{B}_{\mathbf{w}_0}$ (e.g. Manhattan metric in $\mathscr{M} \subset \Omega$, Riemannian metric elsewhere).

Examples:

1. The simplest example is that of a space that is locally Euclidean (*i.e.* equipped with a Euclidean metric everywhere), but is punctured by polygonal obstacles (Figure 2(b)). Due to the 'pointedness' of the obstacles, the constant-g manifolds are essentially circular arcs centered at $\mathbf{q}_0$ or a vertex $\mathbf{v}$ of a polygon. Thus, as illustrated by Figure 2(b), the normals to the arcs are parallel to the tangent to the segment joining $\mathbf{v}$ to $\mathbf{w}_0$.
2. A little less trivial example occurs when the obstacles are not polygonal. Then the statement of the corollary essentially reduces to the assertion that the normal at any point on an involute [1] is parallel to the 'taut string', the end of which traces the involute – and this is true irrespective of the curve used to generate the involute. While the statement has an obvious intuitive explanation by considering the possible directions of motion of the end of the taut string, we provide an explicit computation for an involute created using a circle (Figure 2(c)). Consider a taut string unwrapping off a circle of radius r (starting from $\theta = 0$ when it is completely wrapped). Thus, when the string has unwrapped by an angle θ, the string points at a direction $[\sin(\theta), -\cos(\theta)]^T$. Now, it is easy to verify that the involute is described by the parametric curve $x = r(\cos(\theta) + \theta \sin(\theta)), y = r(\sin(\theta) - \theta \cos(\theta))$. Thus we have $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \theta \cos(\theta)$, $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \theta \sin(\theta)$. Thus the

normal to the involute pointing in the direction $[\frac{\mathrm{d}y}{\mathrm{d}\theta}, -\frac{\mathrm{d}x}{\mathrm{d}\theta}]$ is indeed parallel to the direction in which the string points.

3. The last example that we will illustrate involves a mixture of Manhattan distance (in a particular given coordinate chart) and Euclidean metric. Consider the case in Figure 2(d), where in the given coordinate chart $\mathbf{q}_0$ is the origin, $(0,0)$. For any two points inside the half-plane $\mathscr{M} = \{(x,y) | 2x+y < 2\}$, the distance function is the Manhatan distance. Outside $\mathscr{M}$ it is induced by Euclidean metric. It is to be noted that although the distance function is defined in $\mathscr{M}$, geodesics are not uniquely defined. Let us consider the point $\mathbf{w}_0 = (x,y)$ outside $\mathscr{M}$ (so that there exists a $\mathscr{B}_{\mathbf{w}_0}$ as required by Corollary 1). The distance is given by $d(\mathbf{q}_0, \mathbf{w}_0) = \min_{\alpha\in\mathbb{R}} \left(\alpha + 2(1-\alpha) + \sqrt{(x-\alpha)^2 + (y-2(1-\alpha))^2} \right)$. Denoting the quantity inside 'min' by $f(\alpha)$, and by solving $\frac{\partial f}{\partial \alpha} = 0$, one obtains the unique solution $\alpha = (4x-3y+6)/10$. This gives $d(\mathbf{q}_0, \mathbf{w}_0) = (3x+4y+12)/5$. Thus, the normals to the constant-d surfaces are parallel to $[3/5, 4/5]$. Again, the segments $\overline{\mathbf{v}\mathbf{w}_0}$ have tangent pointing in the direction $[x-\alpha, y-2(1-\alpha)]^T = [(6x+3y-6)/10, (8x+4y-8)/10]^T$. Thus we see that they are indeed parallel.

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