A study on a generalized wave equation for disturbances propagating through a one-dimensional medium placed in a two-dimensional space

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ABSTRACT
In this article mechanical waves propagating through a string restricted to move in a two-dimensional plane has been investigated and a wave equation of a more generalized form has been setup and some interesting observations have been made.

1. INTRODUCTION

Disturbances produced in a string under tension can be described by wave equation. Here we deal with disturbances due to which the particles of the string move in a two-dimensional plane (a plane containing the undisturbed string and a line perpendicular to it).

The wave equation formed with the assumption of i) vertical oscillation of particles of the string, ii) Poisson's ratio = 0 , iii) the oscillations are small, and iv) cross-sectional area and mass per unit length remains constant even when the string is deformed : is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

In this paper we take the consideration of the following points to get a more generalized form of the wave equation:

i) The particles of the string can perform motion in any direction on a two dimensional (x-y) plane. Thus we need a parametric equation to describe the shape of the string at an instant. (Unlike in standard wave equation where y can be expressed as a function of x). Hence, transverse waves ($f_1 = 0$) and longitudinal waves ($f_2 = 0$) can be unified to a pair of generalized wave equations

ii) No assumptions of small oscillations are done. The oscillations can be of any magnitude.

iii) The material has a particular Poisson's ratio '*p*'. When disturbances propagate through the string different tensions exists at different parts of the string. Hence the cross-sectional area of the string changes. Thus the mass per unit length becomes different at different parts of the string.

Notations:

μ_0	Mass per unit length of the un-stretched string,
l	Un-stretched length of the string,
A_0	Cross-sectional area of the un-stretched string,
p	Poisson's ratio for the material,
Y	Young's modulus for the material,
i, **j**	Unit vectors along X and Y directions.

Bold letters denote vectors.
All other mathematical symbols are in *italics*.

List of captions for illustrations:

1. fig-1: Path of motion of a particle initially at a distance ' e ' from one end of the un-stretched string of length ' l '.
2. fig-2: Deformation and displacement of an element of un-stretched length 'de' due to forces.
3. fig-3: Orientation of an element and moment due to forces acting on it.
4. fig-4: Un-stretched string and stretched string with no disturbances propagating through them.

2. SETTING-UP A WAVE EQUATION

2.1. Definitions for starting the problem:

As a disturbance propagates through the string, the particles on the string perform motions. As shown in *fig*-1, the particle initially at a distance ' e ' from one end of the un-stretched string of length ' l ' performs motion along the shown path.

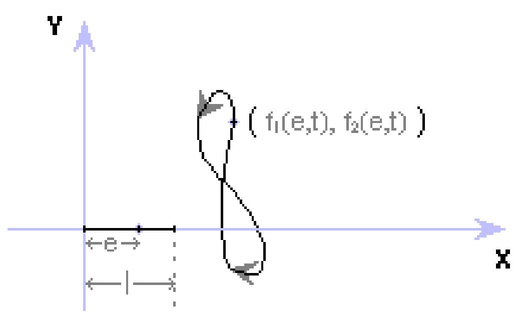

fig-1

The length of initial un-stretched string is ' l '.

Definition of ' e ': A particle of the position ' e ' on the string is defined by the distance of the particle from one end of the un-stretched string.

Let, Position of ' e ' at time ' t ' = (x, y)

$= f(e, t)$

$= (f_1(e, t) , f_2(e, t))$

Hence,

$$\left. \begin{aligned} x &= f_1(e,t) \\ y &= f_2(e,t) \end{aligned} \right\} \qquad (1)$$

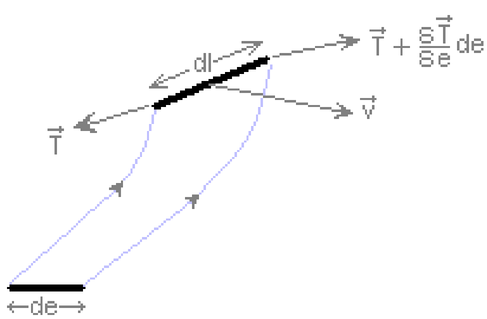

fig-2

A differential element of length ' de ' on the un-stretched string becomes element of length ' dl ' at the position $(x, y) = (f_1, f_2)$ at a particular instant ' t '.

Hence, at that instant,

$$dl = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2} \cdot de \qquad (2)$$

Assuming the validity of Hook's Law,

Stress on the element of un-stretched length ' *de* ' of position ' *e* ' at an instant ' *t* ' (i.e. element at (*x, y*)) is,

$$\begin{aligned}\sigma(e,t) &= Y\cdot(dl-de)/de \\ &= Y\cdot\left(\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2+\left(\frac{\partial f_2}{\partial e}\right)^2}-1\right)\end{aligned} \qquad (3)$$

[Using equation (2)]

where, Y = Young's Modulus

Let Poisson's ratio for the material be ' *p* '.
If ' A_0' is the uniform cross-sectional area of the un-stretched string,
and the cross-sectional. area of the element of un-stretched length ' *de* ' of position ' *e* ' at time ' *t* ' is '*A*', then,

$$p=\frac{\Delta A/A_0}{(dl-de)/de}$$

Hence,

$$\begin{aligned}A &= A_0+\Delta A \\ &= A_0\left(1+\frac{p}{Y}\sigma(e,t)\right)\end{aligned} \qquad (4)$$

[Using equation (3)]

2.2. Determination of tension in the element of un-stretched length ' de ' of position ' e ' at time ' t ':

Tension vector **T**(*e, t*) as shown in *fig-2* is given by,

$$\mathbf{T}=\sigma A\,\mathbf{u} \qquad (5)$$

where, **u** is the unit vector along **T** .

We have already got expressions for *A* and *σ* in terms of *e* & *t*. Now we'll find **u**.

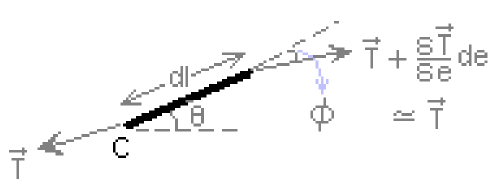

fig-3

As shown in the above figure, **T** makes an angle *Φ* with the tangent at the point.

Considering moment of forces about the point C,

$$\frac{\partial^2\theta}{\partial t^2}=\frac{\tau}{I}$$

where, I = Moment if inertia = $^1/_3$ dm. dl^2,

τ = Torque acting about the point C, and

Again, $\theta=\tan^{-1}\left(\frac{dy}{dx}\right)=\tan^{-1}\left[\left(\frac{\partial f_2}{\partial e}\right)\Big/\left(\frac{\partial f_2}{\partial e}\right)\right]$ = a finite quantity

Therefore,

$$\tau=\frac{\partial^2\theta}{\partial t^2}I$$

$$=\left(\frac{\partial^2\theta}{\partial t^2}\right)\left(\tfrac{1}{3}\,dm\,dl^2\right)$$

$$\therefore\quad \left|\mathbf{T}\times d\mathbf{l}\right|=\left(\frac{\partial^2\theta}{\partial t^2}\right)\left(\tfrac{1}{3}\,dm\,dl^2\right)$$

or, $T\cdot dl\cdot\sin\phi=\left(\frac{\partial^2\theta}{\partial t^2}\right)\left(\tfrac{1}{3}\,dm\,dl^2\right)$

Hence, $\sin\phi=\left(\frac{\partial^2\theta}{\partial t^2}\right)\left(\frac{\frac{1}{3}\,dm\,dl}{T}\right)$

Hence Φ is a differential of order 2 (since the term '*dm dl*' is present in the expression of sin Φ) Therefore, we take $\Phi=0$ for further calculations. i.e. we assume that **T** acts along the tangent of the curve.

Hence,

unit vector along **T** at a particular instant 't'

= unit vector along the tangent at a particular instant 't'

= **u**

$$=(dx\cdot\mathbf{i}+dy\cdot\mathbf{j})\Big/\left(\pm\sqrt{dx^2+dy^2}\right)$$

Hence,

$$\mathbf{u}=\pm\frac{\mathbf{i}+\left(\frac{\partial f_2}{\partial e}\Big/\frac{\partial f_1}{\partial e}\right)\mathbf{j}}{\sqrt{1+\left(\frac{\partial f_2}{\partial e}\Big/\frac{\partial f_1}{\partial e}\right)^2}}\qquad(6)$$

Now, substituting the expressions for σ, A and $\mathbf{u}$ from (3), (4) & (6) into the expression of $\mathbf{T}$ in (5), and doing some simplification, we get the expression for $\mathbf{T}$ as,

$$\mathbf{T} = \pm A_0 Y \cdot \left[1 + p\left(\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2} - 1\right)\right] \cdot \left[1 - 1\Big/\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2}\right] \cdot \left[\frac{\partial f_1}{\partial e}\mathbf{i} + \frac{\partial f_2}{\partial e}\mathbf{j}\right] \qquad (7)$$

For ease of calculations the following substitutions are done,

$$h_1 = 1 + p\left(\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2} - 1\right)$$

$$h_2 = 1 - 1\Big/\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2}$$

$$\mathbf{h_3} = \frac{\partial f_1}{\partial e}\mathbf{i} + \frac{\partial f_2}{\partial e}\mathbf{j}$$

Now differentiating $\mathbf{T}$ w.r.t. e we get,

$$\frac{\partial \mathbf{T}}{\partial e} = \pm A_0 Y\left[h_1 h_2\left(\frac{\partial^2 f_1}{\partial e^2}\mathbf{i} + \frac{\partial^2 f_2}{\partial e^2}\mathbf{j}\right) + h_2 \mathbf{h_3} p\left(\frac{\frac{\partial f_1}{\partial e}\cdot\frac{\partial^2 f_1}{\partial e^2} + \frac{\partial f_2}{\partial e}\cdot\frac{\partial^2 f_2}{\partial e^2}}{\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2}}\right) + \mathbf{h_3} h_1\left(\frac{\frac{\partial f_1}{\partial e}\cdot\frac{\partial^2 f_1}{\partial e^2} + \frac{\partial f_2}{\partial e}\cdot\frac{\partial^2 f_2}{\partial e^2}}{\left(\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2\right)^{3/2}}\right)\right] \qquad (8)$$

2.3. Forming the wave equations:

Please refer to *fig - 2*.

We consider a small element of un-stretched length ' de ' of position ' e '. If the mass per unit length of the un-stretched string be μ_0, then mass of the element is $dm = \mu_0\, de$.

Now, by Newton's Second Law,

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{a}$$

where, $\mathbf{a}$ = acceleration of the centre of mass of the element

$$= \frac{1}{dm}\cdot\frac{\partial \mathbf{T}}{\partial e}\cdot de$$

[$\because$ $\frac{\partial \mathbf{T}}{\partial e} de$ is the resultant force on the element at a particular instant (see *fig-3*)]

Again,

$$\mathbf{v} = \frac{\partial f_1}{\partial t}\mathbf{i} + \frac{\partial f_2}{\partial t}\mathbf{j}$$

Hence, combining the above three we get,

$$\frac{\partial \mathbf{T}}{\partial e} = \mu_0 \left[\frac{\partial^2 f_1}{\partial t^2}\mathbf{i} + \frac{\partial^2 f_2}{\partial t^2}\mathbf{j}\right] \tag{9}$$

where, $\mu_0 = \dfrac{dm}{de}$ is the mass per unit length of the un-stretched string.

Combining (8) & (9) we get ***the wave equation*** **:**

$$\frac{\partial^2 f_1}{\partial t^2}\mathbf{i} + \frac{\partial^2 f_2}{\partial t^2}\mathbf{j} = \pm \mu_0^{-1} A_0 Y \left[h_1 h_2 \left(\frac{\partial^2 f_1}{\partial e^2}\mathbf{i} + \frac{\partial^2 f_2}{\partial e^2}\mathbf{j}\right) + h_2 \mathbf{h_3} p \left(\frac{\frac{\partial f_1}{\partial e}\cdot\frac{\partial^2 f_1}{\partial e^2} + \frac{\partial f_2}{\partial e}\cdot\frac{\partial^2 f_2}{\partial e^2}}{\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2}} \right) + \mathbf{h_3} h_1 \left(\frac{\frac{\partial f_1}{\partial e}\cdot\frac{\partial^2 f_1}{\partial e^2} + \frac{\partial f_2}{\partial e}\cdot\frac{\partial^2 f_2}{\partial e^2}}{\left(\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2\right)^{3/2}} \right) \right] \tag{10}$$

where,

$$h_1 = 1 + p\left(\sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2} - 1\right)$$

$$h_2 = 1 - 1 \Big/ \sqrt{\left(\frac{\partial f_1}{\partial e}\right)^2 + \left(\frac{\partial f_2}{\partial e}\right)^2}$$

$$\mathbf{h_3} = \frac{\partial f_1}{\partial e}\mathbf{i} + \frac{\partial f_2}{\partial e}\mathbf{j}$$

<u>Note</u> that this is a second order vector differential equation. Thus we actually obtain two second order differential equations in which the unknowns are the functions f_1 and f_2. These two functions completely define the motion of the particles on the string.
Thus we attempt to solve for f_1 and f_2.

2.4. Boundary conditions:

i) Initial deformation of the string:

$$f_1(e, 0) = \Omega(e) \qquad \& \qquad f_2(e, 0) = \psi(e)$$

ii) Initial velocity of the particles:

$$\left.\frac{\partial f_1}{\partial t}\right|_{e=e,\, t=0} = \varsigma(e) \qquad \& \qquad \left.\frac{\partial f_2}{\partial t}\right|_{e=e,\, t=0} = \xi(e)$$

iii) String of un-stretched length ' l ' if fixed at two ends at $(0, 0)$ & (L_x, L_y)
i.e. for all t,

$$f_1(0, t) = 0 \qquad \& \qquad f_2(0, t) = 0$$
$$f_1(l, t) = L_x \qquad \& \qquad f_2(l, t) = L_y$$

2.5. Assumptions made while forming the differential equations:

i) For the un-stretched string the mass per unit length(μ_0) is constant.
ii) The cross-sectional area(A_0) of the un-stretched string is constant.
iii) Hook's law is valid for the material in absolute terms,
i.e. Y(young's modulus) = $[T/A]$ / $[\ (l_{\text{final}} - l_{\text{initial}})/l_{\text{initial}}\ ]$

2.6. An attempt to simplify the pair of wave equations:

Equation (10) can easily be discretized and solved numerically using the appropriate boundary conditions. However we try to further simplify the equation by making some approximations.

Approximations made in order to bring the equation (10) close to the standard wave equation:

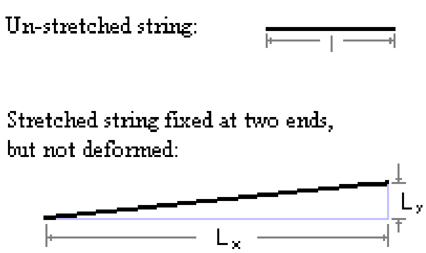

fig-4

i) $L_y << L_x$ [We can easily achieve it by choosing axis suitably]

ii) $f_2 << f_1$ [We can make it possible by allowing only small oscillations]

iii) $\dfrac{\partial f_2}{\partial e} << \dfrac{\partial f_1}{\partial e}$

iv) But $\dfrac{\partial^2 f_2}{\partial e^2} \not\ll \dfrac{\partial^2 f_1}{\partial e^2}$

Justifications behind approximations iii) & iv) :

When $L_y << L_x$, and the oscillations are small, we can express f_1 & f_2 as,

$$f_1(e,t) \approx (L_x / l) \cdot e \ + \ \Delta_1(e,t)$$

$$f_2(e,t) \approx \Delta_2(e,t)$$

where, Δ_1 and Δ_2 are small.

Hence, $\dfrac{\partial f_1}{\partial e} \approx (L_x / l) + \dfrac{\partial \Delta_1}{\partial e}$ & $\dfrac{\partial f_2}{\partial e} \approx \dfrac{\partial \Delta_2}{\partial e}$

And, $\dfrac{\partial^2 f_1}{\partial e^2} \approx \dfrac{\partial^2 \Delta_1}{\partial e^2}$ & $\dfrac{\partial^2 f_2}{\partial e^2} \approx \dfrac{\partial^2 \Delta_2}{\partial e^2}$

From the above it is trivial to conclude that,

$\dfrac{\partial f_2}{\partial e} << \dfrac{\partial f_1}{\partial e}$ and $\dfrac{\partial^2 f_2}{\partial e^2} \not\ll \dfrac{\partial^2 f_1}{\partial e^2}$

With this approximation we get from equation (10) after some simplifications,

$$\frac{\partial^2 f_1}{\partial t^2} = \pm \mu_0^{-1} A_0 Y \left[(1+p)\frac{\partial f_1}{\partial e} - p \right] \cdot \frac{\partial^2 f_1}{\partial e^2} \tag{11a}$$

$$\frac{\partial^2 f_2}{\partial t^2} = \pm \mu_0^{-1} A_0 Y \left[2p\frac{\partial f_1}{\partial e} + 1 - 2p \right] \left[1 - 1 \Big/ \frac{\partial f_1}{\partial e} \right] \cdot \frac{\partial^2 f_2}{\partial e^2} \tag{11b}$$

3. DISCUSSIONS AND CONCLUSIONS:

1. We consider the special case when the elements of the string perform vertical oscillations (i.e. pure transverse waves).

Hence we take $f_1 = k \cdot e$, where $k = L_x / l$

Thus the governing wave equation becomes:

$$\frac{\partial^2 f_2}{\partial t^2} = \pm \mu_0^{-1} A_0 Y [2pk + 1 - 2p][1 - 1/k] \cdot \frac{\partial^2 f_2}{\partial e^2}$$

$$= \pm \mu_0^{-1} A_0 Y [1 + 2p(k-1)][1 - 1/k] k^2 \cdot \frac{\partial^2 f_2}{\partial x^2}$$

[Since, $x = k.e$]

Hence we get,

Velocity of transverse waves in the string is given by,

$$C^2_{transverse} = \mu_0^{-1} A_0 Y\, k[1+2p(k-1)][k-\ 1]$$
$$= \frac{T_u}{\mu} \cdot [1+2p(k-1)]$$

where, T_u = tension in the undisturbed string $\approx A_0 Y.(k - 1)$

μ = mass per unit length of the stretched string = μ_0 / k

2. Now we consider the special case when the elements of the string perform horizontal oscillations (i.e. pure longitudinal waves).

Hence we take, $f_1 \approx k.\ e$, where $k = L_x / l$

And, $f_2 = 0$

Thus the governing wave equation becomes:

$$\frac{\partial^2 f_1}{\partial t^2} = \pm \mu_0^{-1} A_0 Y[k(1+p)-p] \cdot \frac{\partial^2 f_1}{\partial e^2}$$
$$= \pm \mu_0^{-1} A_0 Y[k(1+p)-p]k^2 \cdot \frac{\partial^2 f_1}{\partial x^2}$$

Hence we get,

$$C^2_{longitudinal} = \mu_0^{-1} A_0 Y k^2 [k(1+p)-p]$$
$$= \frac{Y}{\rho_0} \cdot k^2 [k(1+p)-p]$$

where, $\rho_0 = \mu_0 / A_0$ = density of the material of the string in it's un-stretched condition.

With $k = 1$, the values for velocities match exactly with the standard ones.

3. We notice that there is a ± sign in equation (10) [or (11)].

But taking the negative sign gives imaginary values of velocity of waves, provided T_u is positive. i.e. there is tension in the string.

However for negative T_u, i.e. compression, we need to take the negative sign in order to get a real velocity of the wave.

In fact, the origin of the ± sign is from the expression of **T** in equation (8). The two states represent tension and compression.

4. REFERENCES

William T. Thomas, (1975), Theory of Vibration with Application, Prentice Hall.

Irving H. Shames, (1980), Engineering Mechanics, Statics and Dynamics, Prentice Hall.

Egor Popov, (1973), Introduction to Mechanics of Solids, Delhi, Printice Hall.

Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons Inc.