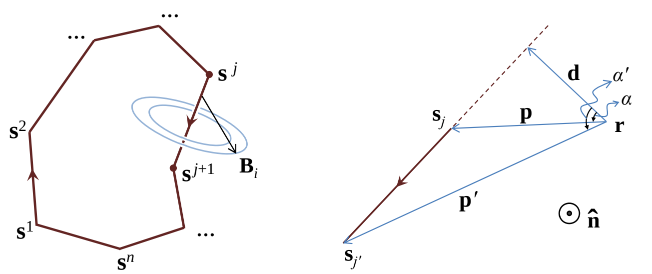
Supplementary material: *h*-signature of a Non-looping Trajectory with Respect to an Infinite Straight Line Skeleton

Subhrajit Bhattacharya
Department of Mechanical Engineering
and Applied Mechanics
University of Pennsylvania
Philadelphia, PA 19104
Email: subhrabh@seas.upenn.edu

Maxim Likhachev
Department of Computer
and Information Science
University of Pennsylvania
Philadelphia, PA 19104
Email: maximl@seas.upenn.edu

Vijay Kumar
Department of Mechanical Engineering
and Applied Mechanics
University of Pennsylvania
Philadelphia, PA 19104
Email: kumar@seas.upenn.edu



(a) A skeleton of an obstacle can be constructed or approximated such that it is made up of n line segments.

(b) Magnetic field at $\mathbf{r}$ due to the current in a line segment $\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}$ can be computed analytically.

Fig. 1.

A. *Recall the definition of* h*-signature*

Definition 1 (*h*-Signature): Given an arbitrary trajectory, τ, in the 3-dimensional environment with M skeletons, we define the *h-signature* of τ to be the following M-vector,

$$\mathcal{H}(\tau) = [h_1(\tau),\ h_2(\tau),\ \ldots,\ h_M(\tau)]^T \tag{1}$$

where,

$$h_i(\tau) = \int_\tau \mathbf{B}_i(\mathbf{l}) \cdot \mathrm{d}\mathbf{l} \tag{2}$$

is defined in an analogous manner as the integral in Ampere's Law. In defining h_i, $\mathbf{B}_i$ is the *Virtual Magnetic Field* vector due to the unit current through skeleton S_i, $\mathbf{l}$ is the integration variable that represents the coordinate of a point on τ, and $d\mathbf{l}$ is an infinitesimal element on τ.

B. *Computation of h-Signature for an Edge of $\mathcal{G}$*

For all practical applications we assume that a skeleton of an obstacle, S_i, is made up of finite number (n_i) of line segments: $S_i = \{\overrightarrow{\mathbf{s}_i^1\mathbf{s}_i^2},\ \overrightarrow{\mathbf{s}_i^2\mathbf{s}_i^3},\ \ldots,\ \overrightarrow{\mathbf{s}_i^{n_i-1}\mathbf{s}_i^{n_i}},\ \overrightarrow{\mathbf{s}_i^{n_i}\mathbf{s}_i^1}\}$ (Figure 1(a)). Thus, the integration of equation (**??**) can be split into summation of n_i integrations,

$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi} \sum_{j=1}^{n_i} \int_{\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}} \frac{(\mathbf{x}-\mathbf{r}) \times \mathrm{d}\mathbf{x}}{\|\mathbf{x}-\mathbf{r}\|^3} \tag{3}$$

where $j' \equiv 1 + (j \mod n_i)$.

One advantage of this representation of skeletons is that for the straight line segments, $\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}$, the integration can be computed analytically. Specifically, using a result from [**?**] (also, see Figure 1(b)),

$$\begin{aligned}\int_{\overrightarrow{\mathbf{s}_i^j\mathbf{s}_i^{j'}}} \frac{(\mathbf{x}-\mathbf{r}) \times \mathrm{d}\mathbf{x}}{\|\mathbf{x}-\mathbf{r}\|^3} &= \frac{1}{\|\mathbf{d}\|}\left(\sin(\alpha') - \sin(\alpha)\right)\hat{\mathbf{n}} \\ &= \frac{1}{\|\mathbf{d}\|^2}\left(\frac{\mathbf{d}\times\mathbf{p}'}{\|\mathbf{p}'\|} - \frac{\mathbf{d}\times\mathbf{p}}{\|\mathbf{p}\|}\right)\end{aligned} \tag{4}$$

where, $\mathbf{d}, \mathbf{p}$ and $\mathbf{p}'$ are functions of $\mathbf{s}_i^j, \mathbf{s}_i^{j'}$ and $\mathbf{r}$ (Figure 1(b)), and can be expressed as,

$$\begin{gathered}\mathbf{p} = \mathbf{s}_i^j - \mathbf{r}\ , \quad \mathbf{p}' = \mathbf{s}_i^{j'} - \mathbf{r}\ , \\ \mathbf{d} = \frac{(\mathbf{s}_i^{j'} - \mathbf{s}_i^j) \times (\mathbf{p}\times\mathbf{p}')}{\|\mathbf{s}_i^{j'} - \mathbf{s}_i^j\|^2}\end{gathered} \tag{5}$$

We define and write $\mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{r})$ for the RHS of Equation (4) for notational convenience. Thus we have,

$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi}\sum_{j=1}^{n_i} \mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{r}) \tag{6}$$

where, $j' \equiv 1 + (j \mod n_i)$.

Given an edge $e \in \mathcal{E}$, we can now compute the *h-signature*, $\mathcal{H}(e) = [h_1(e),\ h_2(e),\ \ldots,\ h_M(e)]^T$, where,

$$h_i(e) = \frac{1}{4\pi}\int_e \sum_{j=1}^{n_i} \mathbf{\Phi}(\mathbf{s}_i^j, \mathbf{s}_i^{j'}, \mathbf{l}) \cdot \mathrm{d}\mathbf{l} \tag{7}$$

can be computed numerically.

Making use of the result from Equation (4), if the current carrying line segment stretches to infinity in both direction (*i.e.* it becomes a line), we have $\alpha' = \frac{\pi}{2}$ and $\alpha = -\frac{\pi}{2}$. The virtual magnetic field due to S_i at a point $\mathbf{r}$ becomes

$$\mathbf{B}_i = \frac{1}{4\pi}\frac{2\ \hat{\mathbf{n}}}{\|\mathbf{d}\|} = \frac{1}{2\pi}\frac{\hat{\mathbf{n}}}{\|\mathbf{d}\|} \tag{8}$$

Note that the contribution of the closing curve at infinity (Construction **??**) becomes zero in the above quantity.

Now consider the straight line segment trajectory $\overline{\tau} = \overline{\mathbf{r}_A\mathbf{r}_B}$. Let the line containing $\overline{\tau}$ (*i.e.* formed by extending $\overline{\tau}$

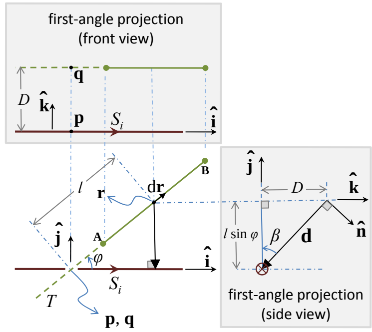


Fig. 2. An infinitely long skeleton and *h*-signature of a straight line segment.

to infinity in both directions) be T (Figure 2). Consider the shortest distance between S_i and T and let it be D. Assuming S_i and T are not parallel, there is a unique point on each of these line ($\mathbf{p}$ and $\mathbf{q}$ respectively) that are closest and are separated by the distance D. The line segment joining the closest points, $\overline{\mathbf{pq}}$, is perpendicular to both S_i and T. The main diagram of Figure 2 shows the projection of S_i and T on a plane perpendicular to $\overline{\mathbf{pq}}$. Note that this plane (the plane of the paper) is parallel to both S_i and T, since it is perpendicular to $\overline{\mathbf{pq}}$.

We define an orthonormal coordinate system with unit vectors $\hat{\mathbf{i}}$ pointing along S_i in the direction of the current, and unit vector $\hat{\mathbf{k}}$ pointing along $\overline{\mathbf{pq}}$. Using these, and referring to Figure 2, we now can write the following equations,

$$\begin{gathered} \|\mathbf{d}\|^2 = D^2 + l^2 \sin^2 \phi \\ \hat{\mathbf{n}} = \cos\beta\ \hat{\mathbf{k}} - \sin\beta\ \hat{\mathbf{j}}\ , \quad \mathrm{d}\mathbf{r} = (\cos\phi\ \hat{\mathbf{i}} + \sin\phi\ \hat{\mathbf{j}})\ \mathrm{d}l \end{gathered} \tag{9}$$

where, ϕ is a constant angle between S_i and T on the plane of the paper, $\cos\beta = \frac{l \sin\phi}{\|\mathbf{d}\|}$, $\sin\beta = \frac{D}{\|\mathbf{d}\|}$, and l is the length parameter along T starting at $\mathbf{q}$. Thus from (8),

$$\mathbf{B}_i \cdot \mathrm{d}\mathbf{r} = -\frac{1}{2\pi} \frac{\sin\beta\ \sin\phi}{\|\mathbf{d}\|} \mathrm{d}l = \ -\frac{D\sin\phi}{2\pi} \frac{\mathrm{d}l}{D^2 + l^2 \sin^2\phi} \tag{10}$$

$$\text{Thus,} \quad \int_{\overline{\tau}} \mathbf{B}_i \cdot \mathrm{d}\mathbf{r} = -\frac{D\sin\phi}{2\pi} \int_{l_A}^{l_B} \frac{\mathrm{d}l}{D^2 + l^2 \sin^2\phi}$$

$$= -\frac{1}{2\pi} \left(\arctan\left(\frac{l_B}{D/\sin\phi} \right) - \arctan\left(\frac{l_A}{D/\sin\phi} \right) \right) \tag{11}$$

An arctangent of a quantity, with consideration for proper quadrants, can assume values between $-\pi$ and π. Thus the quantity within the outer brackets of Equation (11), that is the difference of two arctangents, can assume values between -2π and 2π. Thus the integral $\int_{\overline{\tau}} \mathbf{B}_i \cdot \mathrm{d}\mathbf{r}$, can assume values between -1 and 1. Thus, as claimed in Section **??**, a straight line segment trajectory indeed has the value of $h_i(\tau)$ in $(-1, 1)$ for this simple case of infinitely long line S_i.