Recent Advances in Topological Path Planning

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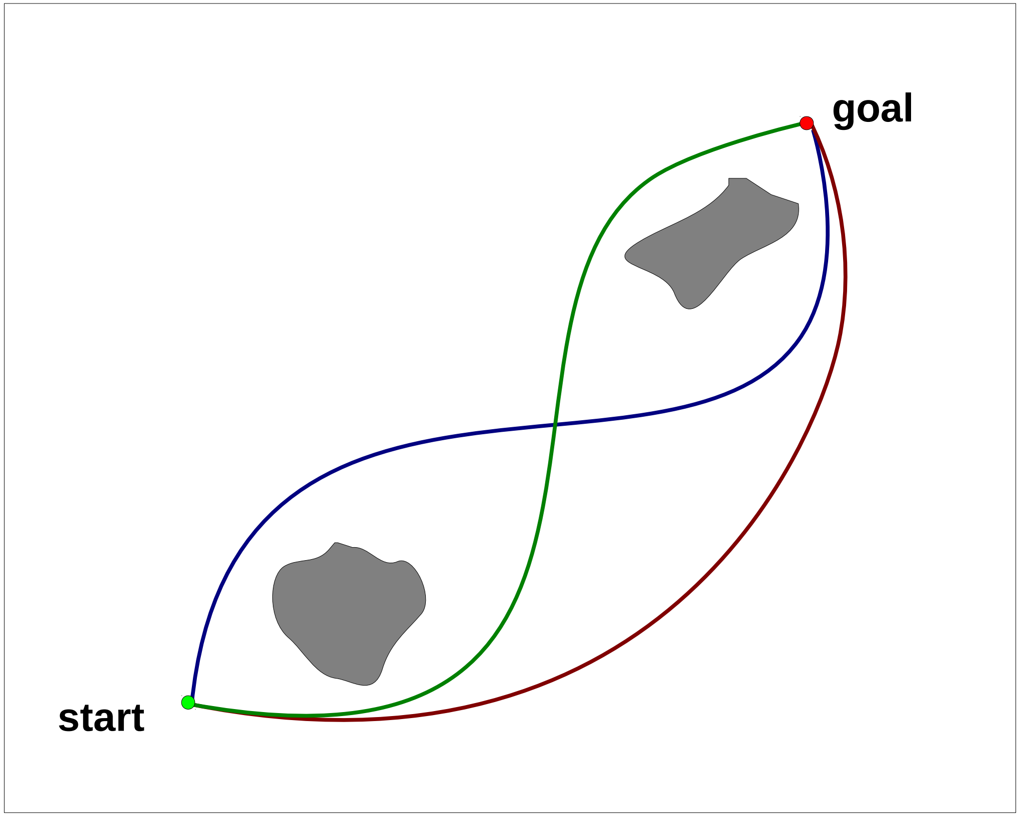
Background

Topological Path Planning

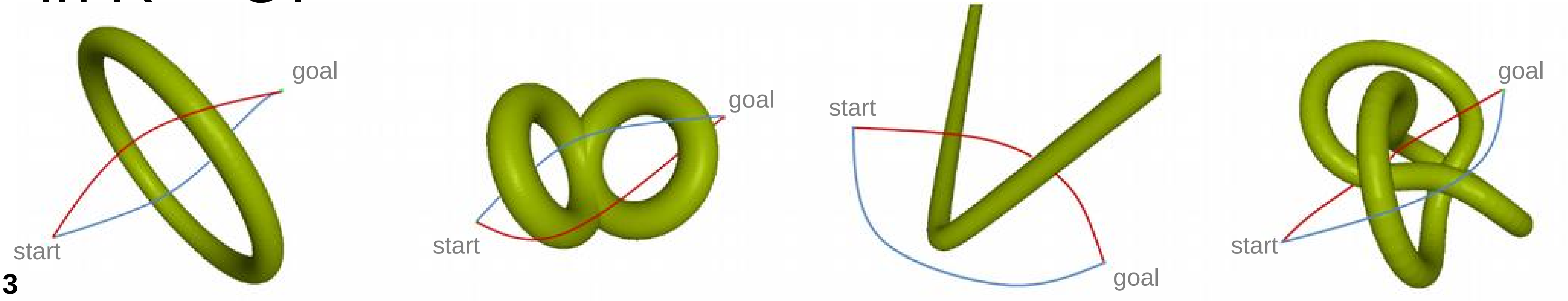
In R^2 - O:



We would like to be able to:

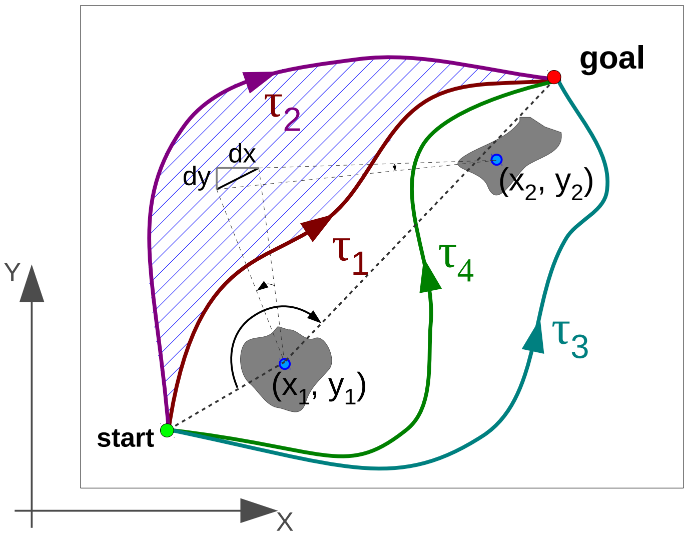
1. Make distinction between the different topological classes of trajectories.
2. Exploit that information for optimal trajectory planning in different topological classes.
3. Apply that to solving real problems in robotics.

In R^3 - O:



Key Idea: "Integrable" Topological Invariants

e.g: Homology Invariants in Planer domain with Obstacles



(x_i,y_i) chosen arbitrarily inside i^{th} obstacle.

$$d\theta_i = \frac{-(y-y_i)\,dx + (x-x_i)\,dy}{\sqrt{(x-x_i)^2 + (y-y_i)^2}}$$

(closed, non-exact forms)

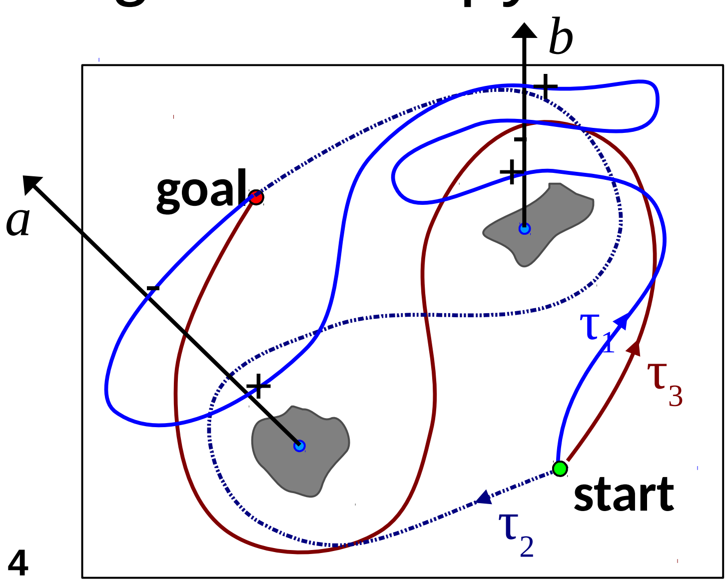
$$\Xi = \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ \vdots \end{pmatrix}$$

One component for each obstacle

$$\int_{\tau_1} \Xi = \int_{\tau_2} \Xi \neq \int_{\tau_4} \Xi \neq \int_{\tau_3} \Xi$$

$$= \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \qquad = \begin{Bmatrix} \alpha - 2\pi \\ \beta \end{Bmatrix}$$

e.g: Homotopy Invariants in Planer domain with Obstacles



Non-intersecting rays.

$h(\tau_1)$ = " b b^{-1} b a a^{-1} "

= " b " (reduced)

$h(\tau_2)$ = " a^{-1} b "

$h(\tau_3)$ = " b a^{-1} "

The words are ***complete homotopy invariants***: Two trajectories are homotopic if and only if their reduced words are the same.

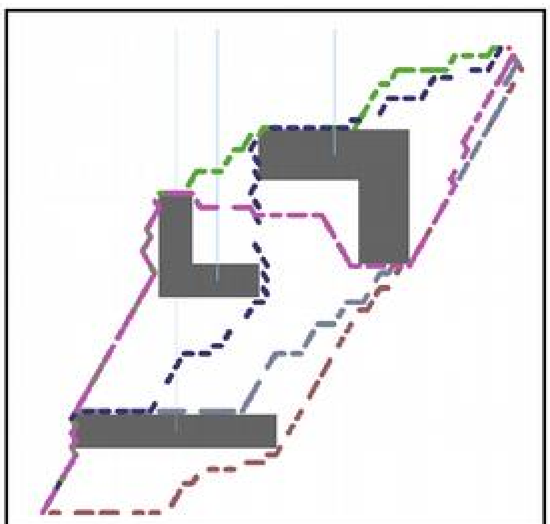
Why? Can we generalize?

But before that, a few robotics applications for motivation...

[Bhattacharya, Likhachev, Kumar. 2012, AURO 33(3)]
[Bhattacharya, Lipsky, Ghrist, Kumar. 2013, AMAI 67(3-4)]

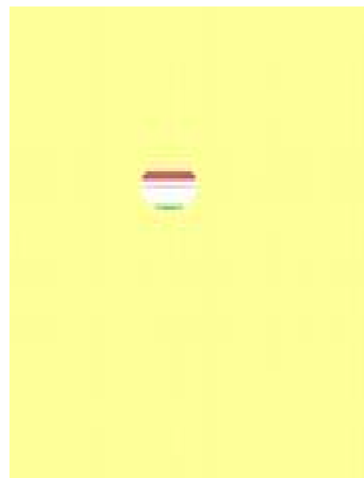
Motivation: Application of Path Homotopy Invariants in Robotics

Can be used in ***breadth-first search algorithms*** (A*, Dijkstra's) for efficiently finding ***optimal paths in different homotopy classes*** (using a graph representation of the configuration space):



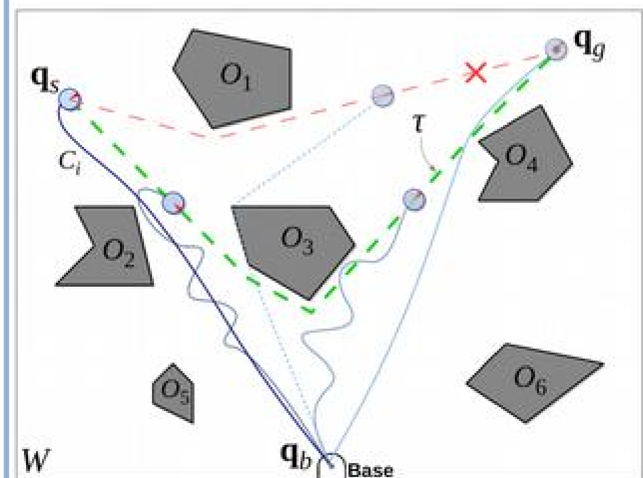
Optimal paths (in graph) in different homotopy classes

Topological multi-robot exploration

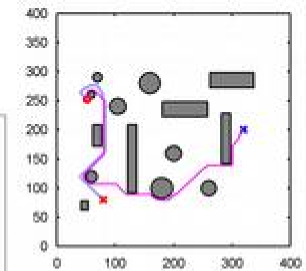


[Kim, Bhattacharya, Ghrist, Kumar. IROS, 2013]

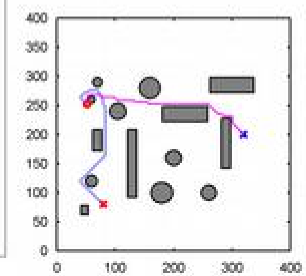
Optimal trajectory planning for tethered robots



[Kim, Bhattacharya, Kumar, ICRA, 2014]

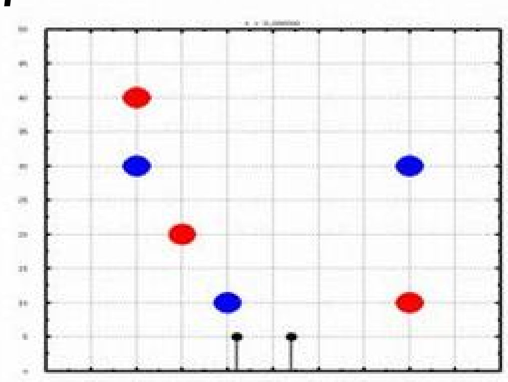


L = 300



L = 450

Topological object separation



[Bhattacharya, Kim, Heidarsson, Sukhatme, Kumar. IJRR, 2014]

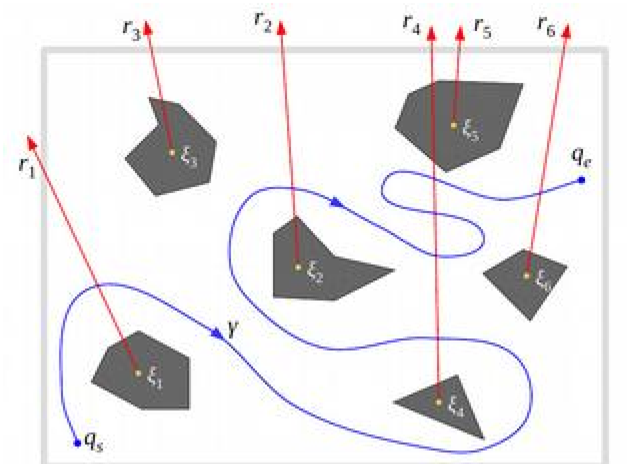
Recent Work

1) Homotopy Invariants in Spatial domain with Obstacles, and their Applications
2) Homotopy Invariants in Coordination Space of Robots Navigating on $\mathbb{R}^2$
3) Application – Path Planning for Cable-controlled Robot
4) Path Planning in high-DOF Systems Through Reeb Graph Construction

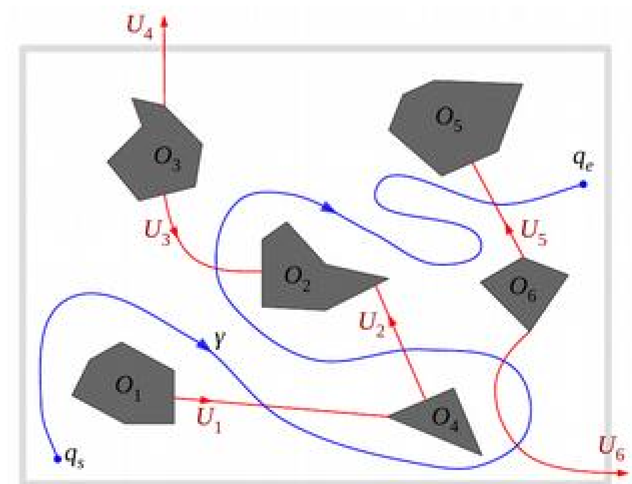
Recent work #1: Homotopy Invariants in Spatial domain with Obstacles, and their Applications

(in collaboration with Robert Ghrist)

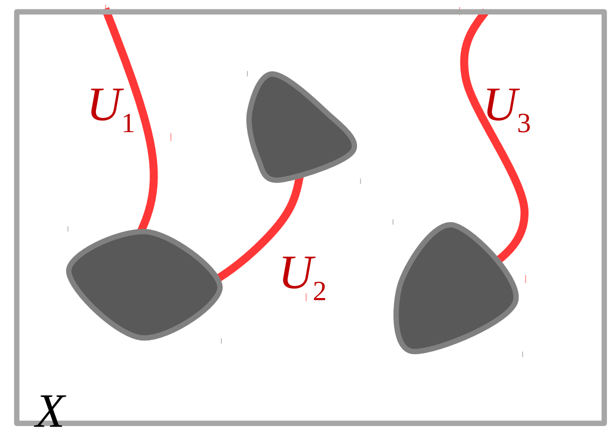
Review of homotopy invariants in planar domains:



$$h(\gamma) = \text{“}r_1^{-1} r_4\, r_2^{-1} r_4^{-1} r_6^{-1}\text{”}$$



$$h(\gamma) = \text{“}u_1^{-1} u_2\, u_3\, u_5^{-1}\text{”}$$



Proposition 1: If

(a) $U_i \cap U_j = \emptyset,\ \forall i \neq j.$
(b) $X - \bigcup_{i=1}^{n} U_i$ *is simply-connected, and,*
(c) $\pi_1(X - \bigcup_{i=1, i\neq j}^{n} U_i) \simeq \mathbb{Z},\ \forall j = 1, 2, \cdots, n,$

(U_i are co-dimension 1 manifolds)

then the “words” are complete path homotopy invariants.

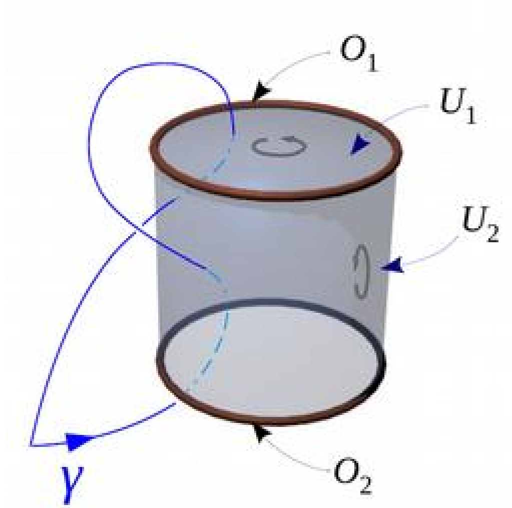
Seifert – van Kampen Theorem:

$$\pi_1(X) \simeq \pi_1(X_0) * \pi_1(X_1) * \cdots * \pi_1(X_n) \simeq *_{i=1}^{n} \mathbb{Z}$$

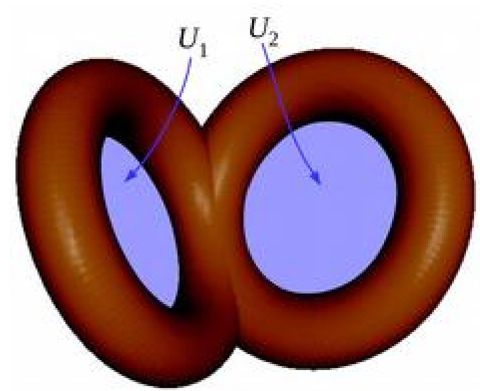
where, $X_j = X - \bigcup_{i=1, i\neq j}^{n} U_i$ $\quad j = 1, 2, \ldots,$

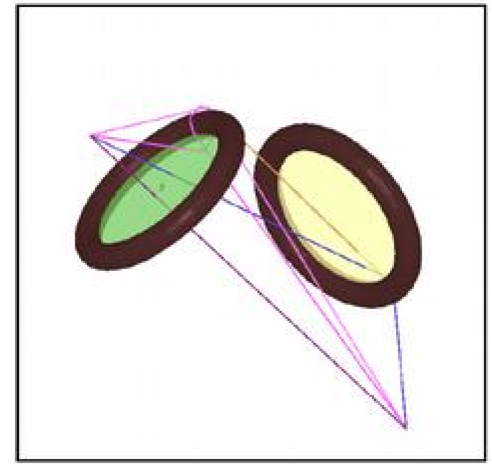
- $\pi_1(X_j) \simeq \mathbb{Z}$ *(represented by ..., u^{-2}, u^{-1}, u^{0}, u^{1}, u^{2}, ...)*
- $X_0 := \bigcap_{i=1}^{n} X_i = X - \bigcup_{i=1}^{n} U_i$ is simply-connected.
  - gives an open cover of X,
  - closed under intersection,
  - any pair-wise intersection giving the simply-connected space, X_0.

In $\mathbb{R}^3$ - O



$h(\gamma)$ = “$u_2\ u_1^{-1}\ u_2^{-1}$”



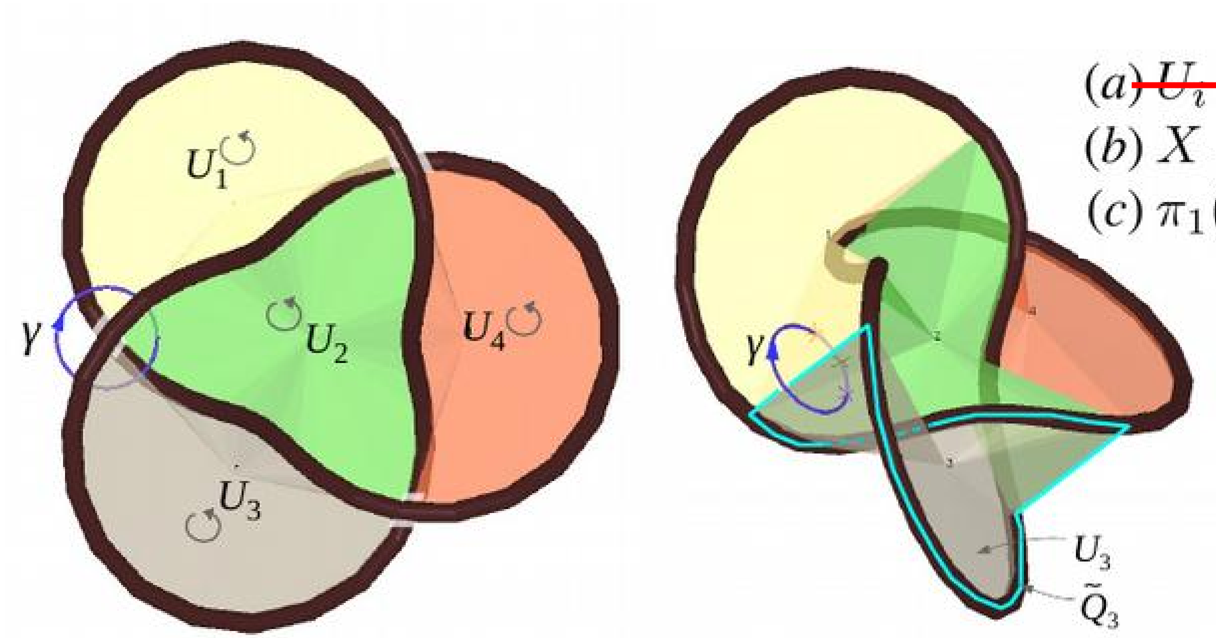


$(a)\ U_i \cap U_j = \emptyset,\ \forall i \neq j.$
$(b)\ X - \bigcup_{i=1}^{n} U_i$ *is simply-connected, and,*
$(c)\ \pi_1(X - \bigcup_{i=1, i\neq j}^{n} U_i) \simeq \mathbb{Z},\ \forall j = 1, 2, \cdots, n,$

→ Proposition 1 holds (words are complete homotopy invariants)

Can’t select such U_i’s if the obstacle is knotted/linked.

Knotted / Linked Obstacles



(a) ~~$U_i \cap U_j = \emptyset,\ \forall i \neq j.$~~
(b) $X - \bigcup_{i=1}^{n} U_i$ *is simply-connected, and,*
(c) $\pi_1(X - \bigcup_{i=1, i\neq j}^{n} U_i) \simeq \mathbb{Z},\ \forall j = 1, 2, \cdots, n$

We still can choose co-dimension 1 manifolds, U_i's, which intersect.

[Other choices, e.g., Seifert surfaces, violate properties (b) or (c), which make things more difficult!]

As a consequence, trivial loops can have non-empty words. *e.g:* $h(\gamma) = \text{“}u_1^{-1} u_2\, u_3^{-1}\text{”}$

- ***Relation set*, R:** The set of all such words obtained from the intersections.
- ***Symmetricized Relation set*, $\overline{\textbf{R}}$:** All the words in R, their inverses and their cyclic permutations.

Siefert – van Kampen Theorem (a more general form):

$$\begin{aligned} \pi_1(X) &\simeq \pi_1(X_0) * \pi_1(X_1) * \cdots * \pi_1(X_n) \ / N \\ &\simeq *_{i=1}^{n} \mathbb{Z} \ / N \end{aligned}$$

where, N is the normal subgroup generated by the words corresponding to the trivial loops (including all their cyclic permutations and inverses).

Algorithm: Dehn's Algorithm for Word Problem

Problem: Given the symmetricized Relation set, $\overline{R}$,

Check if two paths, τ_1 and τ_2, connecting the same points, belong to the same homotopy class. [Equivalently, whether or not $w := h(\tau_1 \cup -\tau_2) \in N$.]

Various ***algorithms*** exist, each with ***completeness & termination guarantees*** only for ***specific classes of groups and presentations***.

Dehn's metric algorithm:

1. Cyclically reduce w.
2. For every $\rho \in \overline{R}$, check if w and ρ share a subword of length < $|\rho|/2$. If yes, replace the subword in w with the smaller "equivalent".

Until no further reduction possible.

Ex.: Suppose $w = h(\tau_1 \cup -\tau_2)$ =

"$u_4\ u_3\ u_2\ u_2^{-1} u_6^{-1} u_1\ u_3\ u_4^{-1}$"

"$u_3\ u_6^{-1}\ u_1\ u_3$"

Assuming, "$u_6^{-1} u_1\ u_5$" $\in \overline{R}$,

"$u_3\ u_5^{-1}\ u_3$"

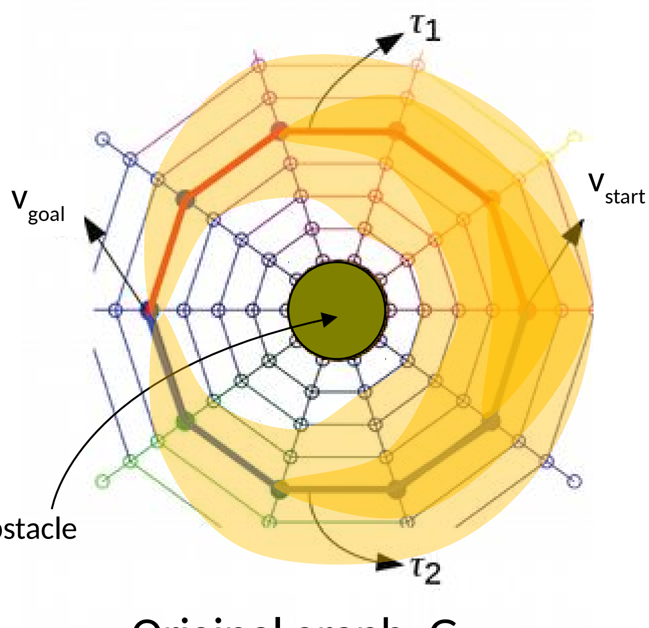
Properties:

Low computational complexity, always terminates, paths are homotopic if terminates to empty word, but…

if it does not terminate to empty word, the paths are not necessarily not homotopic (except under certain specific circumstances).

Algorithm: Use in Graph Search

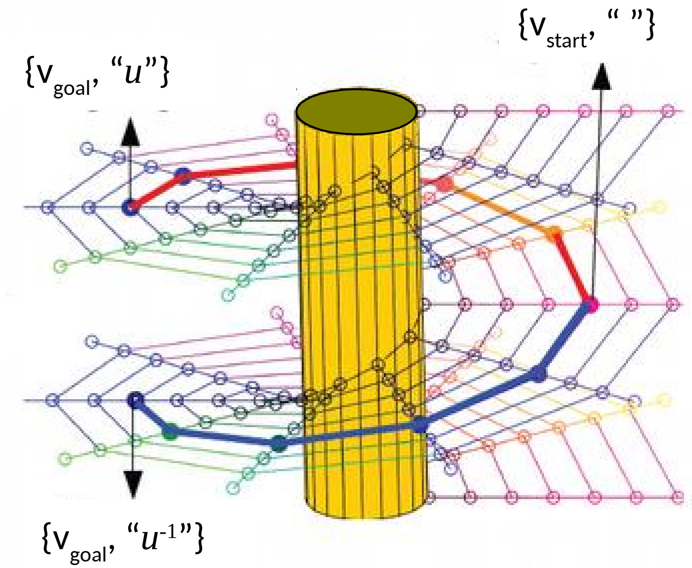
Illustration in 2-D case:



Original graph, G

Graph search algorithm

h-augmented graph:

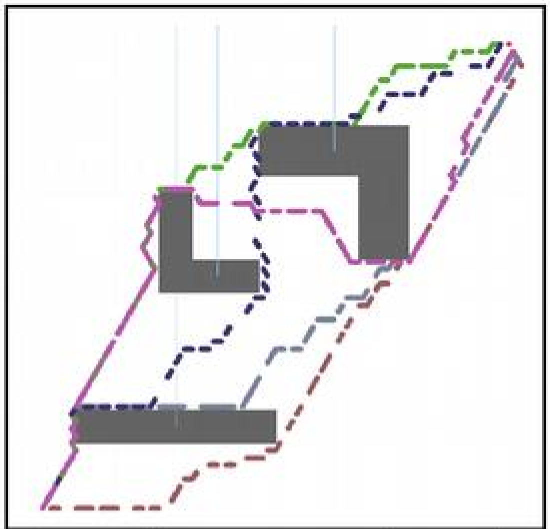


h-augmented graph, G_h

(Graph representation of the universal cover of the configuration space)

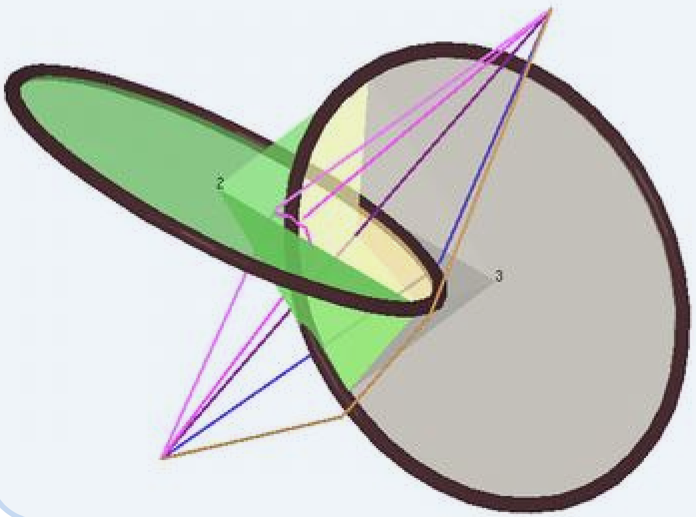
Lets us compute ***optimal trajectories*** (in the discrete graph) in ***different homotopy classes*** using a single run of the search algorithm. Trajectories are generated i***n order of path length***.

Example in $\mathbb{R}^2$ - O

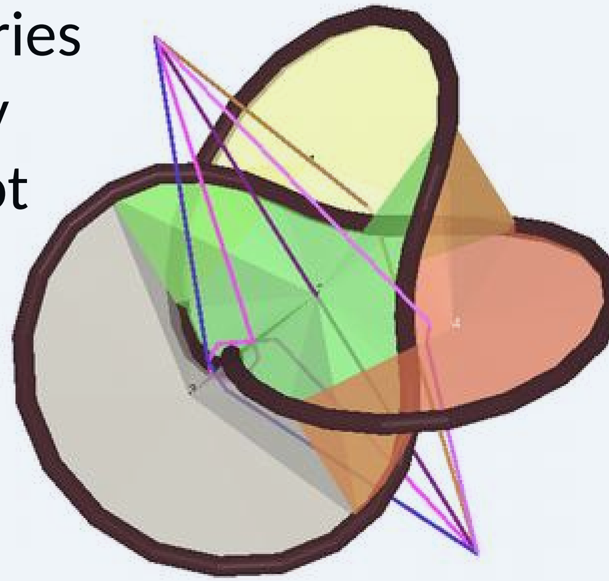


Results in $\mathbb{R}^3$- O

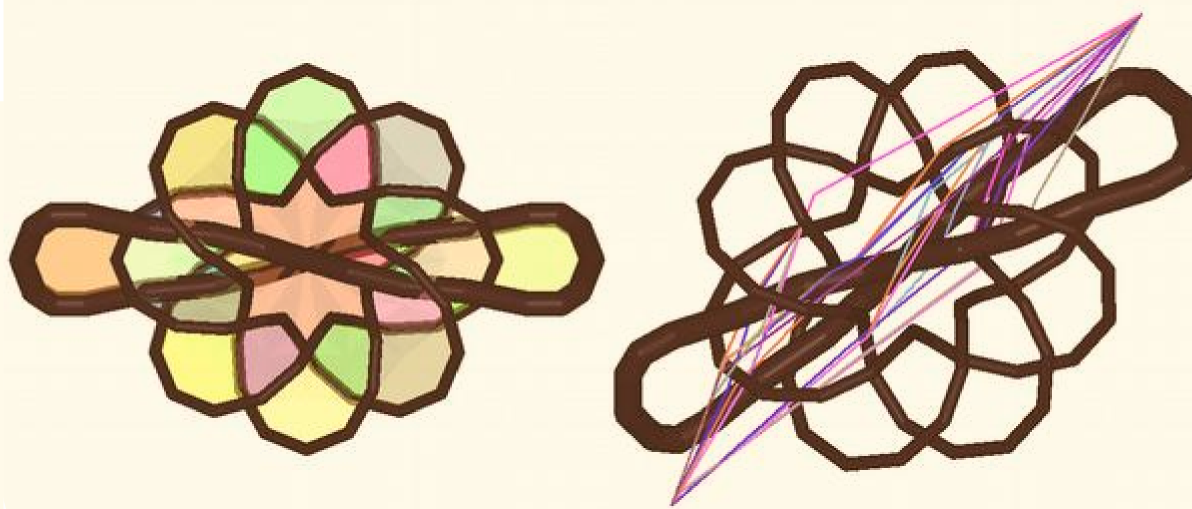
Five shortest trajectories in different homotopy classes in a Hopf link complement



Five shortest trajectories in different homotopy classes in a trefoil knot complement



20 shortest trajectories in different homotopy classes in the complement of a (3, 8) torus knot linked to a genus-2 torus



Recent work #2: Homotopy Invariants in Coordination Space of Robots Navigating on $\mathbb{R}^2$

(in collaboration with Robert Ghrist)

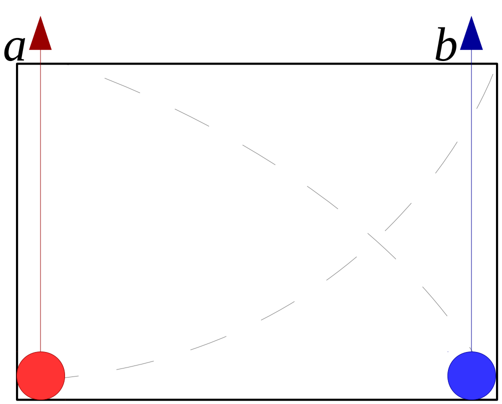
Individual robot's configuration space: Γ_i

Cylindrically deleted coordination space:

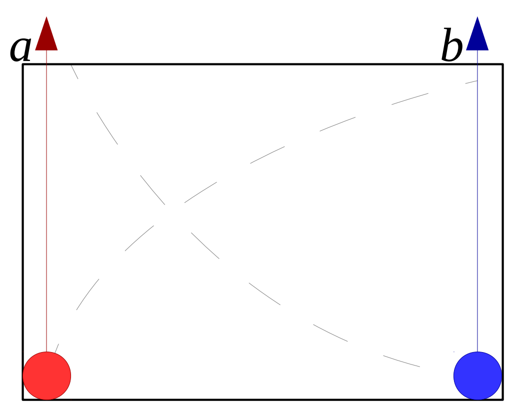
$$X = \underbrace{\left(\prod_{i=1}^{n} \Gamma_i \right)}_{\overline{\Gamma}} - \mathcal{O} \quad \text{where} \quad \mathcal{O} = \bigcup_{i<j} \underbrace{\left\{ (x_k)_1^N \in \prod_{k=1}^{N} \Gamma_k \ : \ (x_i, x_j) \in \Delta_{i,j} \right\}}_{\mathcal{O}_{i,j}}$$

"Cylindrical" obstacles: $\mathcal{O}_{i,j} = \Delta_{i,j} \times \left(\prod_{k \neq i,j} \Gamma_k \right)$

where, pairwise collision sets: $\Delta_{i,j} \subset \Gamma_i \times \Gamma_j$, with $1 \leq i < j \leq N$



Robot 'b' crosses the ray of 'a'.



Robot 'a' crosses the ray of 'b'.

Co-dimension 1 candidates for U_*:

$$\mathcal{U}_{i,j} = \{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid x_i = x_j, y_i < y_j\}$$

Boundary is $\mathcal{O}_{i,j}$.
But these can intersect with other $\mathcal{O}_{i',j'}$ and $\mathcal{U}_{i',j'}$.

Homotopy Invariants in Coordination Space of Robots Navigating on $\mathbb{R}^2$

Need to carefully partition $\mathcal{U}_{i,j}$ so that the partitions satisfy the conditions of Proposition 2. Also, need to find the corresponding relation set, $\mathbf{R}$.

The final choice for the co-dimension 1 manifolds, U_*:

$$\mathcal{U}_{m,p/\sigma_{m+1},\sigma_{m+2},\cdots,\sigma_{p-1}} =$$

$$\left\{ (x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \mid y_m < y_p, \begin{pmatrix} x_m = x_p \leq x_n \text{ if } \sigma_n = \text{`}-\text{'} \\ x_m = x_p \geq x_n \text{ if } \sigma_n = \text{`}+\text{'} \end{pmatrix}, \forall\, m < n < p \right\}$$

For every possible choice of σ_* = ‘+’ or ‘-’

Relation set contains words of the form:

$$\text{“}u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}} \cdot u^{-1}_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u^{-1}_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}}\text{”}$$

$$\text{“}u_{i,j/\sigma_{i+1},\cdots,\sigma_{j-1}} \cdot u_{i',j'/\gamma_{i'+1},\cdots,\gamma_{j'-1}} \cdot u^{-1}_{i,j/\sigma_{i+1},\cdots,\sigma_{j-1}} \cdot u^{-1}_{i',j'/\gamma_{i'+1},\cdots,\gamma_{j'-1}}\text{”}$$

$$\text{“}u_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot u^{-1}_{m,n/\alpha_{m+1},\cdots,\alpha_{n-1}} \cdot u^{-1}_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}}.$$

$$\text{“}u_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(1)}=-,\cdots,\sigma_{p-1}} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}} \cdot u^{-1}_{m,p/\sigma_{m+1},\cdots,\sigma_n^{(2)}=+,\cdots,\sigma_{p-1}} \cdot u_{n,p/\beta_{n+1},\cdots,\beta_{p-1}}\text{”}$$

Homotopy Invariants for Cylindrically Deleted Coordination space

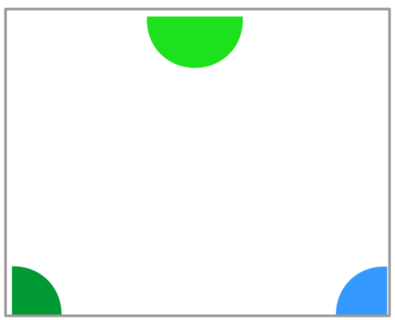
(for point robots navigating on the Euclidean plane)

Example: $N = 3$

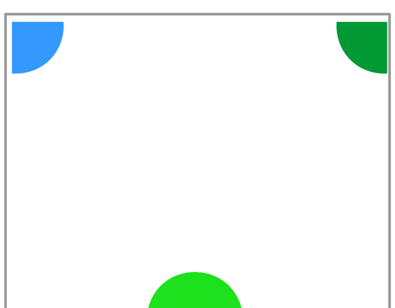
$$\mathcal{U}_{1,2} = \{\mathbf{p} \mid x_1 = x_2, y_1 < y_2\},$$
$$\mathcal{U}_{2,3} = \{\mathbf{p} \mid x_2 = x_3, y_2 < y_3\}$$
$$\mathcal{U}_{1,3/+} = \{\mathbf{p} \mid x_1 = x_3 > x_2, y_1 < y_3\}$$
$$\mathcal{U}_{1,3/-} = \{\mathbf{p} \mid x_1 = x_3 < x_2, y_1 < y_3\}$$

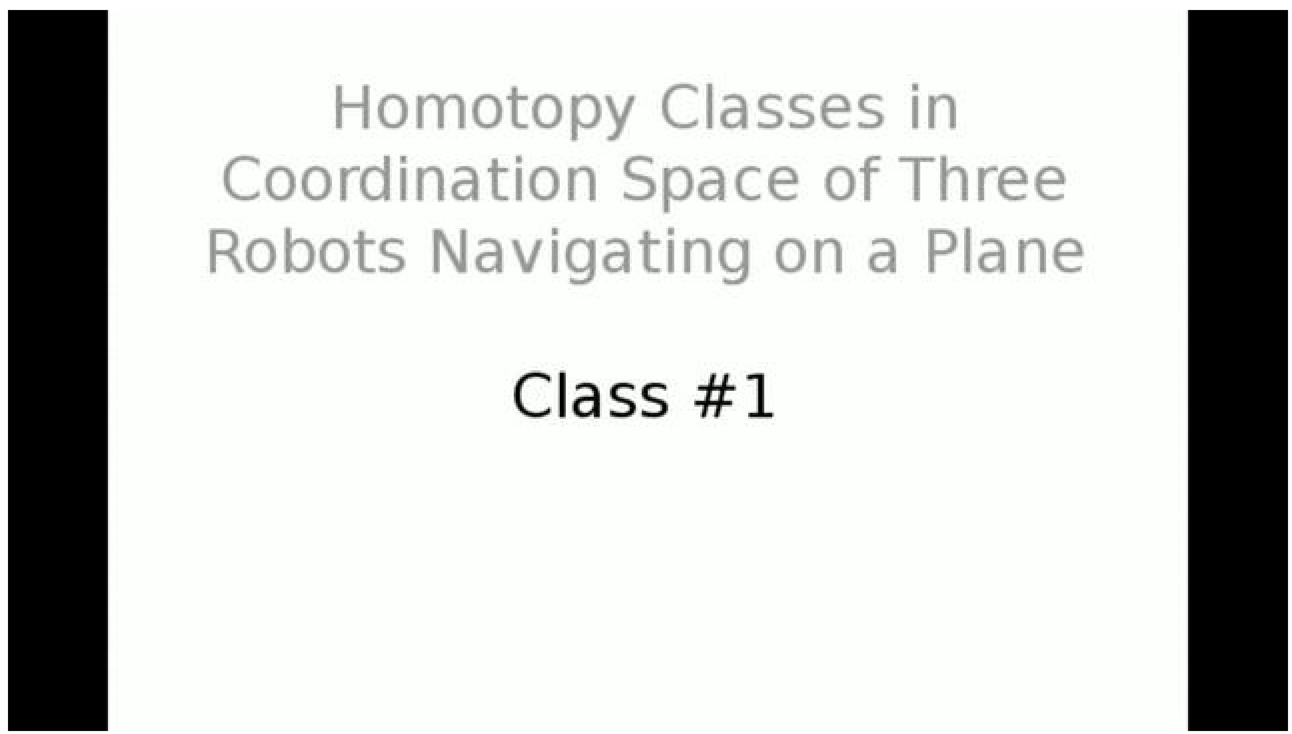
$$\mathbf{R} = \{\ u_{1,2}\ u_{1,3/-}\ u_{2,3}\ u_{1,2}^{-1}\ u_{1,3/+}^{-1}\ u_{2,3}^{-1},\ u_{1,2}\ u_{1,3/-}\ u_{1,2}^{-1}\ u_{1,3/+}^{-1},\ u_{1,3/-}\ u_{2,3}\ u_{1,3/+}^{-1}\ u_{2,3}^{-1}\ \}$$

Start



Goal



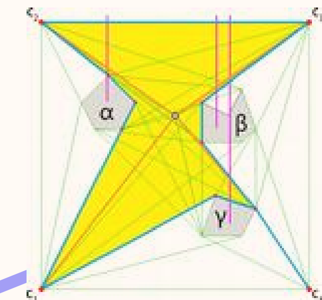


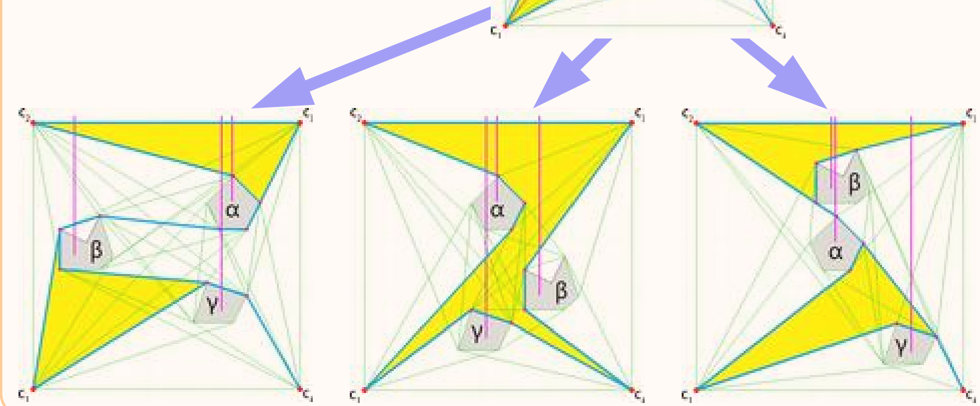
Recent work #3: Application – Path Planning for Cable-controlled Robot

*(done with **Xiaolong Wang**)*

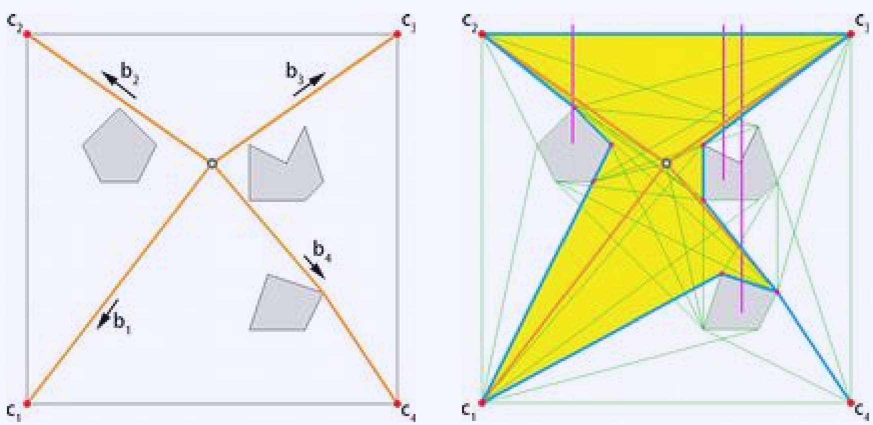
Skycam, source:
Wikimedia Commons

Workspace with Moving Obstacles – Maximizing Expected Area

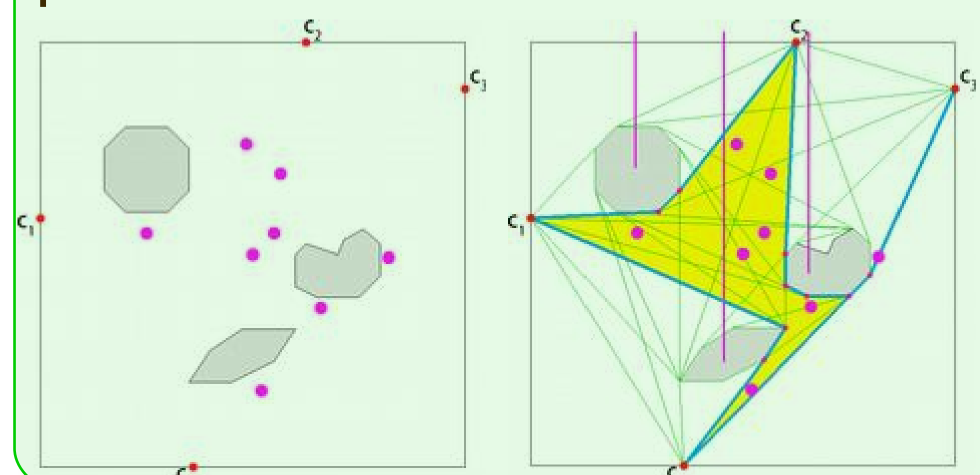




Compute workspace's boundary and area from initial cable configuration



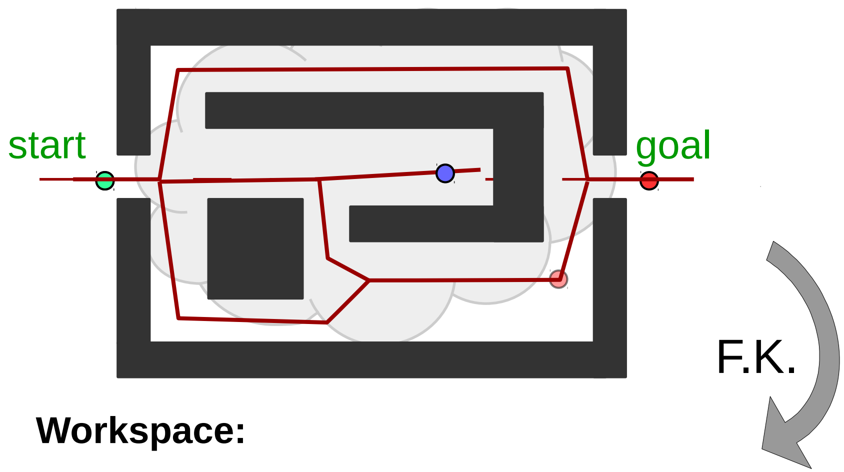
Maximize workspace covering multiple task points



Recent work #4: Path Planning in high-DOF Systems Through Reeb Graph Construction

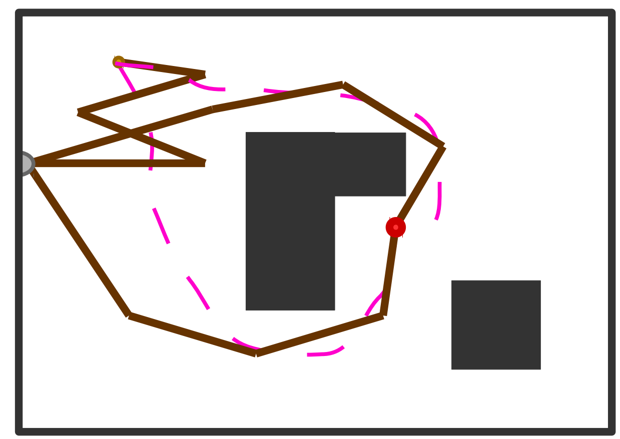
(in collaboration with Mihail Pivtoraiko)

Config. Space:



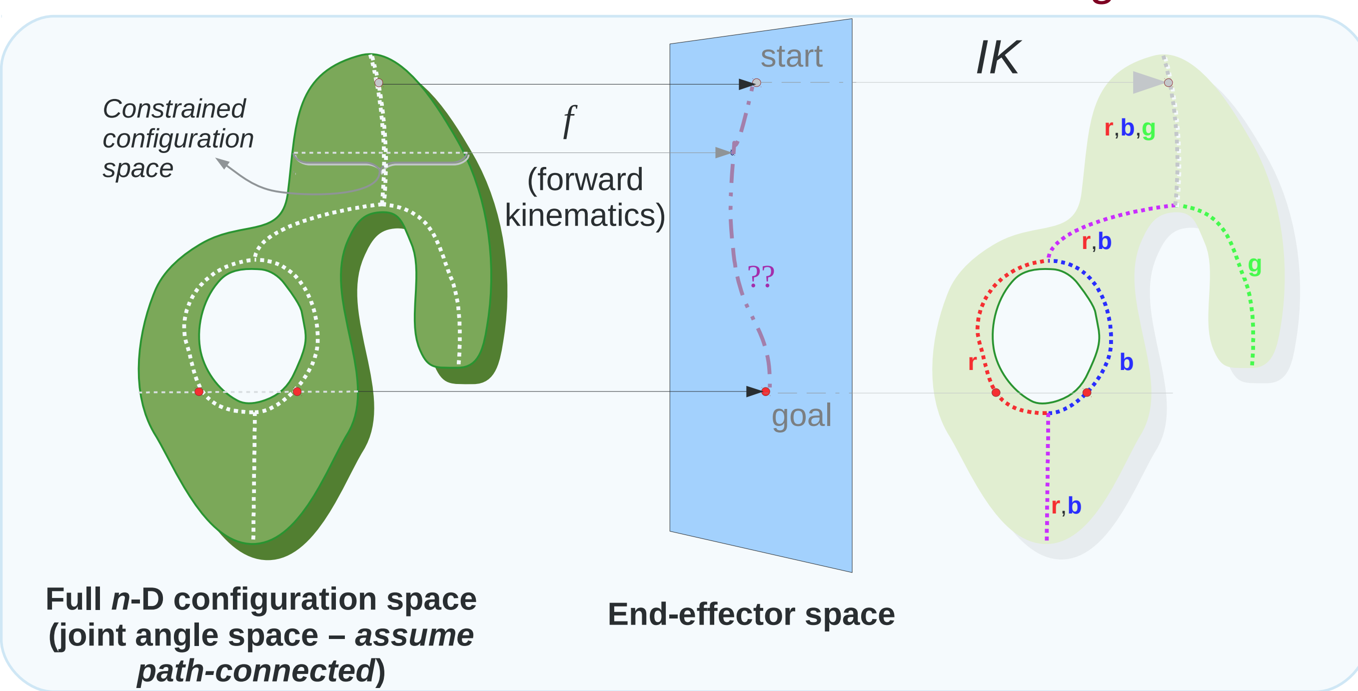
Workspace:

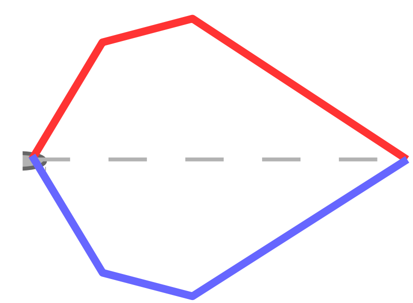
Example: Robotic Manipulator



- Take end-effector to a desired target location
- Optimization of trajectory of end-effector (*e.g.*, its length)

(We do not care where the rest of the arm is, as long as it does not intersect an obstacle!)





Even if the workspace of a planar arm is obstacle-free, the Reeb graph in the config. space is non-trivial (the structure of which vary greatly based on the arm geometry)

Completely classified!

Conclusions

Topological Path Planning: The use of topological invariants in conjunction with discrete search-based algorithms in order to:

- Create lower-dimensional abstractions/equivalents of configuration spaces for reduction of computational complexity
- Compute solutions in distinct topological classes

Direct applications to systems involving flexible cables and articulated systems.

Thank you!

Questions?