Topological complexity of surfaces

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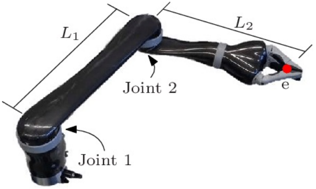
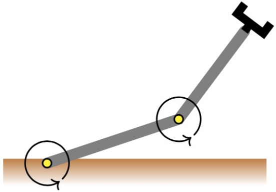
Configuration space





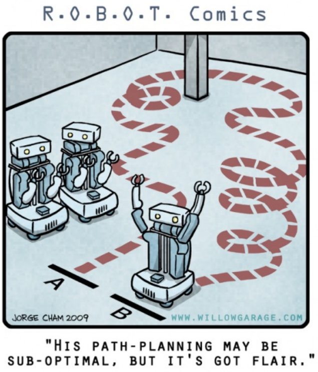
Configuration space of a mechanical system is the topological space X whose points parameterize possible configurations/positions

Assumptions:
X is path connected
X has the homotopy type of a finite CW-complex

Example

2 joint planar robot arm has configuration space
$X = \{(\theta_1, \theta_2)\} = S^1 \times S^1 = T$ the torus

Motion planning problem

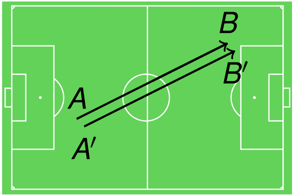


Find an algorithm which, given configurations A, B in X, produces a motion of the system from A to B

Rephrase: Given $A, B \in X$, exhibit $\gamma = \gamma(A, B) \colon I \to X$ $\quad I = [0, 1]$ continuous path $\gamma(t) = \gamma(A, B)(t)$ from $A = \gamma(0)$ to $B = \gamma(1)$

Motion planning . . .

. . . is sometimes easy



can globally exhibit paths depending continuously on input (A, B)

. . . becomes more involved as the topology of X gets interesting

cannot globally exhibit paths depending continuously on input (A, B)

Motion planning . . .

$PX = \{\gamma\colon I \to X \text{ continuous}\}$ free paths on X (compact/open topology)
$\mathrm{ev}_{0,1}\colon PX \to X \times X \qquad \mathrm{ev}_{0,1}(\gamma) = (\gamma(0), \gamma(1))$

Motion planning problem: find $s\colon X \times X \to PX$ with $\mathrm{ev}_{0,1} \circ s = \mathrm{id}_{X\times X}$

Call such s a *motion planner*

Desirable: motion planner depends continuously on input (A, B)

Fact: $\exists s\colon X \times X \to PX$ continuous with $\mathrm{ev}_{0,1} \circ s = \mathrm{id}_{X\times X} \iff X \simeq *$

so in general we divide $X \times X$ into pieces

motion planner $s\colon X \times X \to PX$ is *tame* if $X \times X = F_1 \cup F_2 \cup \cdots \cup F_k$

1. $F_i \cap F_j = \emptyset$ for $i \neq j$
2. Each restriction $s_i = s|_{F_i}\colon F_i \to PX$ is continuous
3. Each F_i is nice - a Euclidean neighborhood retract

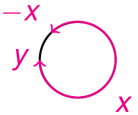
Tame motion planner solves motion planning problem:
input $(A, B) \in X \times X \qquad \exists! F_i$ with $(A, B) \in F_i \qquad$ output $s_i(A, B)(t)$

Topological complexity of X TC(X)

$\mathrm{TC}(X) = \min\{k \mid \exists$ tame motion planner with $X \times X = F_1 \cup \cdots \cup F_k\}$

Example $\mathrm{TC}(\mathbb{R}^n) = 1 \quad \mathbb{R}^n \times \mathbb{R}^n = F_1 \quad s_1(A,B)(t) = (1-t)\cdot A + t \cdot B$

Example ($X = S^1$)

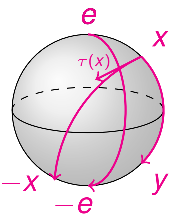


$F_1 = \{(x,-x) : x \in X\} \subset X \times X$
$F_2 = X \times X \smallsetminus F_1 = \{(x,y) : x \neq -y\}$

$s_1 = s|_{F_1} : F_1 \to PX$ counterclockwise path from x to $-x$
$s_2 = s|_{F_2} : F_2 \to PX$ shortest geodesic arc from x to y $\quad \mathrm{TC}(S^1) = 2$

Example ($X = S^2$)

fix $e \in X$, τ a nowhere zero tangent vector field on $X \smallsetminus e$



$F_1 = \{(e,-e)\}$
$F_2 = \{(x,-x) : x \neq e\}$
$F_3 = \{(x,y) : x \neq -y\}$

$s_1 = s|_{F_1} : F_1 \to PX$ any fixed path from e to $-e$
$s_2 = s|_{F_2} : F_2 \to PX$ path x to $-x$ along semicircle tangent to $\tau(x)$
$s_3 = s|_{F_3} : F_3 \to PX$ shortest geodesic arc from x to y $\quad \mathrm{TC}(S^2) \leq 3$

Homotopy invariance

Farber (2004) $\mathrm{TC}(X)$ is the *sectional category* or *Schwarz genus* of the fibration $\mathrm{ev}_{0,1}\colon PX \to X \times X, \gamma \mapsto (\gamma(0), \gamma(1))$

$$\mathrm{TC}(X) = \min \left\{ k \;\middle|\; \begin{array}{l} X \times X = U_1 \cup \cdots \cup U_k \quad \text{with } U_i \text{ open and} \\ \exists s_i\colon U_i \to PX \text{ continuous section } \mathrm{ev}_{0,1} \circ s_i = \mathrm{id}_{U_i} \end{array} \right\}$$

Homotopy invariance of $\mathrm{TC}(X)$ easier to see from this perspective

Determining $\mathrm{TC}(X)$ can be difficult

Farber-Tabashnikov-Yuzvinsky (2003)

If $n = 1, 3,$ or 7 $\quad \mathrm{TC}(\mathbb{RP}^n) = 1 + n$

If $n \neq 1, 3, 7$ $\quad \mathrm{TC}(\mathbb{RP}^n) = 1 + \text{immersion dimension of } \mathbb{RP}^n$

In particular $\mathrm{TC}(\mathbb{RP}^2) = 4$

motion planning in $\mathbb{RP}^2$ $\qquad \mathbb{RP}^2 \times \mathbb{RP}^2 = U_a \cup U_{xy} \cup U_{xz} \cup U_{yz}$

$U_a = \{(A, B)\}$ lines through 0 in $\mathbb{R}^3$ which make an acute angle
$s_a\colon U_a \to P(\mathbb{RP}^2)$ rotate A to B through this angle

$U_{xy} = \{(A, B)\}$ lines whose projections onto the xy-plane span
$s_{xy}\colon U_{xy} \to P(\mathbb{RP}^2)$ rotate A to B via plane orientation induced by (A, B)

Bounds

Dimensional bound $\mathrm{TC}(X) \leq 2 \cdot \dim(X) + 1$ $\quad X \simeq$ finite CW-complex

Tame motion planner with $2n + 1$ elements $\quad n = \dim(X)$

$X^k = k$-skeleton of X $\quad S_k = X^k \smallsetminus X^{k-1}$ union of interiors of k-cells

$F_i = \bigcup_{k+\ell=i} S_k \times S_\ell$ disjoint ENRs with $F_0 \cup \cdots \cup F_{2n} = X \times X$

define $s_i \colon F_i \to PX$ by defining s_i on $S_k \times S_\ell$ $\quad k + \ell = i$

Product inequality $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$

$\{F_i, s_i\}_{i=1}^n$ and $\{F'_j, s'_i\}_{j=1}^m$ tame motion planners for X and Y

$\implies \{F''_k, s''_k\}$ tame motion planner for $X \times Y$

$$F''_k = \bigcup_{i+j=k} F_i \times F'_j \qquad s''_k = s_i \times s'_j \text{ on } F_i \times F'_j \qquad 2 \leq k \leq n + m$$

Cohomology lower bound $\mathrm{TC}(X) > \mathrm{zcl}\, H^*(X; \Bbbk)$ zero-divisor cup length

$$\mathrm{zcl}\, H^*(X; \Bbbk) = \text{cup length}\big(\ker(H^*(X; \Bbbk) \otimes H^*(X; \Bbbk) \xrightarrow{\cup} H^*(X; \Bbbk))\big)$$

$A = \bigoplus_{m \geq 0} A^m$ graded algebra $\quad$ *cup length*$(A) = \mathrm{cl}(A)$ is the biggest q such that $\exists a_1, \ldots, a_q \in A^{>0}$ (homogeneous) with $a_1 \cdots a_q \neq 0$

sectional category(fibration $p \colon E \to B$) $> \mathrm{cl}\big(\ker(p^* \colon H^*(B; \Bbbk) \to H^*(E; \Bbbk))\big)$

Topological complexity of orientable surfaces

Example ($X = S^2$ continued)

tame motion planner $S^2 \times S^2 = F_1 \cup F_2 \cup F_3 \implies \mathrm{TC}(S^2) \leq 3$

if $0 \neq x \in H^2(S^2; \Bbbk)$ then $(x \otimes 1 - 1 \otimes x)^2 = -2x \otimes x \neq 0$

$\mathrm{zcl}\, H^*(S^2; \Bbbk) \geq 2 \implies \mathrm{TC}(S^2) \geq 3$ $\mathrm{TC}(S^2) = 3$

Example ($X = S^1 \times S^1 = T$)

Product inequality $\implies \mathrm{TC}(S^1 \times S^1) \leq \mathrm{TC}(S^1) + \mathrm{TC}(S^1) - 1 = 3$

if $H^1(S^1 \times S^1; \Bbbk) = \langle u, v \rangle$ then $(u \otimes 1 - 1 \otimes u)(v \otimes 1 - 1 \otimes v) \neq 0$

$\mathrm{zcl}\, H^*(S^1 \times S^1; \Bbbk) \geq 2 \implies \mathrm{TC}(S^1 \times S^1) \geq 3$ $\mathrm{TC}(S^1 \times S^1) = 3$

Example ($X = \Sigma_g = T\# \cdots \# T$ orientable genus $g \geq 2$)

Dimensional inequality $\implies \mathrm{TC}(\Sigma_g) \leq 5$

If $H^1(\Sigma_g; \Bbbk) = \langle u_i, v_i \mid 1 \leq i \leq g \rangle$ then

$(u_1 \otimes 1 - 1 \otimes u_1)(v_1 \otimes 1 - 1 \otimes v_1)(u_2 \otimes 1 - 1 \otimes u_2)(v_2 \otimes 1 - 1 \otimes v_2) \neq 0$

$\mathrm{zcl}\, H^*(\Sigma_g; \Bbbk) \geq 4 \implies \mathrm{TC}(\Sigma_g) \geq 5$ $\mathrm{TC}(\Sigma_g) = 5$

Topological complexity of non-orientable surfaces

Example ($X = \mathbb{RP}^2$) Farber-Tabashnikov-Yuzvinsky

$\mathsf{TC}(\mathbb{RP}^2) = 1 + \text{immersion dimension of } \mathbb{RP}^2 = 4$

Example ($X = N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ non-orientable genus $h \geq 2$)

$$\left.\begin{array}{rcl} \text{Dimensional inequality} & \Longrightarrow & \mathsf{TC}(N_h) \leq 5 \\ \operatorname{zcl} H^*(N_h;\mathbb{Z}_2) \geq 3 & \Longrightarrow & \mathsf{TC}(N_h) \geq 4 \end{array}\right\} \Longrightarrow 4 \leq \mathsf{TC}(N_h) \leq 5$$

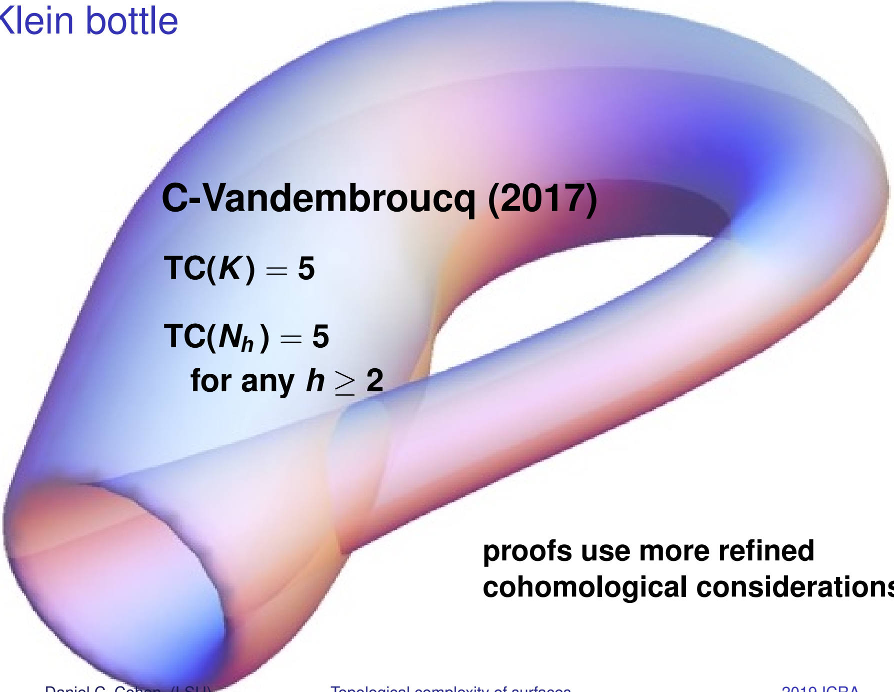
Dranishnikov (2016)

- $\mathsf{TC}(N_h) = 5$ for $h \geq 4$ obstruction theory $\mathsf{TC}(M\#N) \geq \mathsf{TC}(N)$ for M, N surfaces with M orientable...
- shows that his approach do not extend to N_h for $h = 2, 3$

so... $\mathsf{TC}(N_h) = \mathsf{TC}_2(N_h) =?$ for $h = 2, 3$

in particular what is $\mathsf{TC}(K)$? $K =$ Klein bottle

Klein bottle



C-Vandembroucq (2017)

TC(K) $=$ 5

TC(N_h) $=$ 5
for any $h \geq 2$

proofs use more refined cohomological considerations