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Galois theory. (English)

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Galois theory is often described as one of the most fascinating and beautiful parts of mathematics. Its historical roots can be traced back to the first attempts to find solutions of cubic and quartic polynomial equations in ancient times, and the brilliant but not fully developed ideas of Évariste Galois in the early 19th century gave birth to many of the fundamental concepts of modern abstract algebra, including groups and fields. In a sense, Galois theory typifies the genesis of modern mathematics in its consistency, ranging from Évariste Galois's first discoveries up to Alexandre Grothendieck's visionary programs in contemporary algebra and geometry.
Being such a jewel of mathematics, Galois theory is one of the high-lights of every university course in advanced algebra. According to both its fundamental character and its significance in modern algebra, number theory, and geometry, Galois theory can be taught on different levels of abstraction and generality, and this very fact is reflected by the meanwhile huge number of distinct textbooks on the subject. The book under review is another one in the row of those texts that grew out of the author's courses on classical, basically elementary Galois theory for graduate students. Its main goal is to develop classical Galois theory from scratch, requiring of the reader only the basic facts about vector spaces, groups, and polynomial rings, but nevertheless discussing the subject systematically and in considerable generality, together with many illustrating examples and concrete applications.
In contrast to other comparable textbooks on classical Galois theory, the present one does not follow the strictly historical path starting from ancient geometric construction problems and their related algebraic equations, which by the way is motivating and enlightening enough, but focuses instead on E. Artin's very elegant approach emphasizing linear algebra [*E. Artin*, Galois theory (University of Notre Dame Press, Notre Dame) (1942; Zbl 0060.04813), reprint by (Dover Publications, New York) (1998; Zbl 1053.12501)] which quickly leads to the fundamental theorems in their modern setting. Otherwise, the text offers the standard material of classical field theory and Galois theory, though in a remarkably original, unconventional and comprehensive manner, with many further-leading remarks and examples.
As to the precise contents, the book consists of five chapters and three appendices. However, strangely enough, there are no bibliographic references or hints for further reading.
Chapter 1, representing already one of the peculiar features of the book under review, gives a purely informal introduction to Galois theory by a number of typical examples, even before the theoretical concepts have been properly set up. Of course, this introductory chapter is didactically motivated, pointing to the destination of what is to come in the sequel.
Chapter 2 develops the basics of fields and field extensions up to the Fundamental Theorem of Galois Theory. Assuming no prior knowledge of field theory, the author discusses generalities on fields, the algebraic properties of polynomial rings, field extensions, splitting fields, normal extensions, separable extensions, and Galois extensions. Then, in the concluding section of this chapter, the fundamental theorem of Galois theory is proven à la E. Artin, that is by using characters of Galois groups. Numerous concrete examples are given to illustrate the theory developed so far, and more than 20 exercises complete this chapter.
Chapter 3 explains more advanced aspects and applications of the Galois correspondence, including symmetric functions and the symmetric group, separability criteria for

field extensions, the structure of finite fields, disjoint extensions, simple extensions, the normal basis theorem for Galois extensions, abelian extensions, and the rudiments of Kummer theory. This chapter concludes with a section on the norm and the trace map for finite Galois extensions, and exactly 30 exercises at the end complement the material of this part of the book.
Chapter 4 is devoted to algebraic number fields and their related Galois theory, together with applications to the celebrated ancient problems that eventually led to the subject in the abstract. The variety of topics treated here incorporates the following: factorization and irreducibility in $\mathbb{Q}[X]$, cyclotomic polynomials and fields, solvable extensions and groups, constructibility by ruler and compass, quadratic number fields, radical extensions, Galois groups of number fields, and the discriminant of a polynomial. This chapter concludes with a section on practical computations of Galois groups, followed by another set of nearly 30 carefully selected exercises.
The final Chapter 5 deals with a few more advanced topics in Galois theory, including also infinite field extensions. A finer analysis of separable and inseparable field extensions, normal extensions and Galois closures, the algebraic closure of a field, and the generalization of the fundamental theorem of Galois theory to infinite Galois extensions form the sections of this chapter, complemented by a few (mainly theoretical) exercises. As the Krull topology is the key to understanding infinite Galois extensions, only in this final section of the book the reader is assumed to have the necessary knowledge of general topology.
There are three appendices providing some auxiliary (or additional) material from basic group theory, commutative algebra, and elementary number theory related to the main text. These appendices are added for logical and pedagogical reasons, just in order to keep the line of arguments concerning Galois theory uninterrupted.
Altogether, the book under review must be seen as a highly welcome and valuable complement to existing textbook literature on classical Galois theory. It comes with its own features and advantages, reflecting the author's glaring passion and experience as a university teacher, and it surely is a perfect introduction to this evergreen subject. The numerous explaining remarks, hints, examples, and applications are particularly commendable from the instructional point of view, just as the outstanding clarity and fullness of the text.

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Keywords : general field theory; textbook; algebraic equations; field extensions; algebraic number fields; polynomials
Classification :
∗12-01 Textbooks (field theory)
12F10 Galois theory
11R32 Galois theory for global fields
12E12 Algebraic equations
12E20 Finite fields (field-theoretic aspects)
11R11 Quadratic extensions