UNIVERSAL SUPERHARMONIC FUNCTIONS AND THEIR APPLICATION TO THE CONFORMAL TYPE OF PROPER MINIMAL SURFACES IN $\mathbb{R}^3$

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1. Introduction

The material in all but the last section of this abstract can be found in [3], especially section 6. Many of the underlying ideas can be traced back to [1].

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2. Basic notions

Underlying any minimal surface is a Riemann surface. This is part of the Weierstrass data, and the determination of the underlying Riemann surface of a minimal surface, either exactly or in terms of some broad class, is an important problem in the field.

Recall that harmonic functions are invariant under a conformal change of metric, so the question of whether or not a surface admits a non-constant, bounded harmonic function depends only on the conformal structure of the manifold. We refer to this question as the question of the conformal type of the surface (note that this term is often used for other, similar properties).

There are a few ways of introducing Brownian motion on a manifold; intuitively, we think of it as the continuous version of an isotropic random walk on a manifold. More precisely, it solves the martingale problem for half the Laplacian; that is, if B_t is Brownian motion and f is smooth and compactly supported, then $f(B_t) - f(x_0) - \int_0^t \frac{1}{2}\Delta f(B_s)\, ds$ is a martingale. (See [5] or [2] for more background on Brownian motion on Riemannian manifolds.) Brownian motion on a surface M is called *recurrent* if any of the following equivalent conditions hold:

- There exists an open, precompact set $A \subset M$ and a point $x \in M$ (with $x \notin \overline{A}$) such that Brownian motion started at x almost surely hits A.
- For any $x \in M$ and open, precompact $A \subset M$, Brownian motion started at x almost surely hits A.
- Brownian motion returns infinitely often to any (equivalently, some) open, precompact A, almost surely.

If M is not recurrent, it is *transient*. In this case, Brownian motion almost surely has a last time in any compact set. We note that recurrence and transience depend only on the conformal structure of M.

Our next task is to consider the relationship between bounded harmonic functions and Brownian motion. If M is recurrent, M admits no non-constant bounded

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harmonic functions. On the other hand, if M is transient, it may or may not admit a non-constant bounded harmonic function. Fortunately, we when consider surfaces with non-empty boundaries, the natural notions for bounded harmonic functions and Brownian motion are equivalent. In particular, a surface M with non-empty boundary ∂M is *parabolic* if any of the following equivalent conditions hold:

- Any bounded harmonic function on M is determined by its boundary values on ∂M.
- There exists a point $x \in M$ such that Brownian motion started from x hits ∂M almost surely.
- BM started from any point hits ∂M almost surely.

Note that if a surface M is recurrent or parabolic (depending on whether ∂M is empty), then M with a compact set added or removed is also recurrent or parabolic.

3. Universal superharmonic functions

Recall that a function is *superharmonic* if its Laplacian is everywhere non-positive. It is well known that if a surface M with non-empty boundary admits a positive, proper superharmonic function, then M is parabolic.

Definition 1. *Let U be a non-empty, open subset of $\mathbb{R}^3$. A function $f : U \to \mathbb{R}$ is a* universal superharmonic function on U *if the restriction of f to any minimal surface (possibly with boundary) in U is superharmonic.*

For example the x_i are universal superharmonic functions on all of $\Re^3$. More interestingly, let $r = \sqrt{x_1^2 + x_2^2}$. Then, for any minimal surface M,

$$|\Delta_M \log r| \leq \frac{|\nabla_M x_3|^2}{r^2} \quad \text{on } M \setminus \{x_3\text{-axis.}\}$$

It follows that

- $\log r - x_3^2$ is a universal superharmonic function on $\left\{r \geq 1/\sqrt{2}\right\}$
- $\log r - x_3 \arctan x_3 + \frac{1}{2} \log\left(x_3^2 + 1\right)$ is a universal superharmonic function on $\left\{r \geq \sqrt{1 + x_3^2}\right\}$

Consider the slab $S(C) = \{0 \leq x_3 \leq C\}$ for some $C > 0$. Using that $\log r - x_3^2 + C^2$ is a proper, positive, superharmonic function on any properly immersed minimal surface contained in $S(C) \cap \{r \geq 1\}$, we outline the proof of (see Theorem 6.7 of [3] and Theorem 3.1 of [1])

Theorem 2. *Let M be a properly immersed minimal surface, possibly with boundary, contained in $\{x_3 \geq 0\}$. If $\partial M = \emptyset$, then $M = \{x_3 = c\}$ for some $c \geq 0$. If $\partial M \neq \emptyset$, then M is parabolic. In particular, if a properly immersed minimal surface (without boundary) intersects any plane in a compact set, it is recurrent.*

4. A more geometric application: area growth

We now wish to see how the universal superharmonic function f $f = \log r - x_3^2$ can be used to control the growth of the area of a properly immersed minimal surface-with-boundary, which is contained in a slab. Such a surface arises when considering certain ends of properly embedded minimal surfaces. In particular, we sketch the proof of the fact that such a minimal surface-with-boundary, which we denote E, has quadratic area growth. That is,

$$\int_{E\cap\{r\leq t\}} dA = Ct^2 + o\left(t^2\right).$$

The argument relies on the divergence theorem and the relationship between Δf and two more geometric quantities, namely $|\nabla x_3|^2$ and $\Delta \log r$.

We note that analogous results, namely parabolicity and quadratic area growth, can be proven for minimal surfaces-with-boundary contained between two half-catenoids, rather than contained in a slab, by using the "other" universal superharmonic function mentioned above, namely

$$f = \log r - x_3 \arctan x_3 + \frac{1}{2}\log\left(x_3^2+1\right).$$

Recall that a minimal surface is contained between two half-catenoids if $|x_3| \leq C\log r$ for large r.

5. More Brownian motion

We again consider a properly immersed minimal surface M, possibly with boundary, contained in the halfspace $\{x_3 \geq 0\}$. This time we wish to use Brownian motion to understand bounded harmonic functions. Using that x_3 composed with Brownian motion on M is a martingale, we give an alternative proof of Theorem 2 (see the proof of Theorem 2.2 of [4] for the basic approach). In particular, the argument seems to rely on the same underlying structure as the proof mentioned above, but doesn't make any use of universal superharmonic functions.

In light of this last point, it might be of some interest to better understand the relationship between these two approaches. For example, is there a similar Brownian motion-based proof of the analogous result for a minimal surface contained between two half-catenoids? More generally, does Brownian motion encode other information about ends of properly embedded minimal surfaces contained between two plane or half-catenoids?

References

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