Coupling efficiency between light pipes of different dimensions

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An expression for the transmitted fraction of the uniform and parallel radiation incident upon the coupler between two rectangular light pipes of different physical dimensions is derived. It is a function of the pipe dimension ratio and the angle of the incident radiation. It is shown that for radiation nearly parallel to the pipe axis, it is possible to design a coupler for any pipe dimension ratio so that all the radiation incident upon it is transmitted by it. Design curves for various incidence angles and the pipe dimensions are presented.

I. Introduction

Light pipes with specularly reflecting walls are useful for optical information processing, spatial filtering, and optical communications. The propagation characteristics of these pipes were first reported by Poehler[1] in 1970. His results were later extended by Powell[2] to include the skew rays and by Wagh and Rao[3] to include the right angle bends in these pipes. In this paper, coupling between two light pipes of different dimensions is investigated. An explicit relation for the fraction of the incident radiation that is transmitted by the coupling is derived, assuming perfectly reflecting walls.

Long gradual couplers are generally expected to perform better than short couplers. This is shown to be true only in the case of incidence radiation nearly parallel to the optical axis. For large incidence angles, shorter couplings are at times better than the long ones. Design curves for various light pipe dimension ratios and incident radiation angles are presented.

II. Statement of the Problem

Consider the coupling between two rectangular light pipes of dimensions $2a \times c$ and $2b \times c$ as shown in Fig. 1. Both the pipes and the coupling are assumed to have specularly reflecting walls. The open end of the larger tube is uniformly illuminated with parallel rays whose direction is defined by angles θ and ϕ as shown in Fig. 1. The coupling is completely specified by the dimensions $2a$, $2b$, c, and the angle ψ made by the tilted faces of the coupling with the optical axis. The problem of coupling design can now be stated as follows: Given two light pipes of dimensions $2a \times c$ and $2b \times c$ and radiation directions ϕ and θ to find ψ so that the maximum amount of radiation in the larger tube can go into the smaller tube.

We define the efficiency of coupling, η as the fraction of the incident radiation transmitted by the coupler. In this paper we want to derive the condition for η to be 1.

It is obvious from Fig. 1 that neither the angle θ nor the reflections from the two parallel faces of the coupling affect the forward direction of a light ray from the larger tube to the smaller tube. Similarly the dimension c and the coordinate along this dimension of the point of entry of the ray into the tube merely determine the coordinates of the exit point of the ray. The angles ψ and ϕ, the dimensions a and b, and the entrance coordinate along dimension a of the ray, on the other hand, decide whether it is going to pass through the coupling or not. For the purpose of determining coupling efficiency it is therefore sufficient to deal with the projection of this structure along dimension c. This projection and the coordinate system used are shown in Fig. 2. Efficiency of the coupler η is, then, the range of y_0, the rays within which go through the coupler divided by the total range of y_0, $2a$.

The length of the light pipe before the coupling is not a factor that will in any way influence the coupling efficiency. Thus, we shall ignore the light pipe before the coupling and assume that the $z = 0$ plane of the coupling is itself illuminated uniformly by parallel rays.

III. Analysis

We denote the angle made by a ray with the z axis after i reflections in the coupling by $\phi_i (0 \leq \phi_i \leq \pi)$.

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Received 28 August 1975.

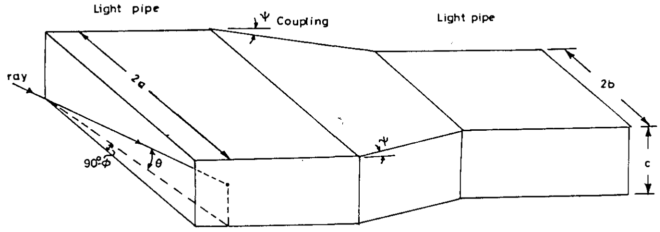

Fig. 1. Coupling between two light pipes of dimensions $2a \times c$ and $2b \times c$ and the angles θ and ϕ defining the direction of incident radiation.

Clearly $\phi_0 = \phi$ and from the geometry of the structure, $\phi_i = -(\phi_{i-1} + 2\psi)$. This makes $\phi_i = (-1)^i(\phi + 2i\psi)$ and gives the maximum number of reflections $n_{\max}$ of the ray before it is reflected back toward the larger tube

$$(\text{from } |\phi_{n_{\max}+1}| > \pi/2, |\phi_{n_{\max}}| \le \pi/2)$$

as

$$n_{\max} = [(\pi - 2\phi)/4\psi], \tag{1}$$

where $[x]$ denotes the largest integer not larger than x. Moreover, a ray that has turned toward the larger tube cannot turn back again because the magnitude of its angle with the z axis increases at every reflection by 2ψ until $|\phi_i| > \pi - \psi$, in which case it will go out of the larger end of the coupling without any more reflections if it has not gone out already.

To find out if a ray starting from the $z = 0$ plane will be able to cross the coupling, it is therefore sufficient to check the ray's ability to cross the length of the coupling in $n_{\max}$ reflections.

We now find (y_{n+1},z_{n+1}) the coordinates of the point of the $(n + 1)$th reflection of a ray whose nth reflection is at (y_n,z_n). If (y_{n+1},z_{n+1}) is on S_1 and (y_n,z_n) on S_2, from the equation of the ray and the surfaces S_1 and S_2,

$$(y_{n+1} - y_n) = \tan\phi_n(z_{n+1} - z_n),$$

$$(a - y_{n+1}) = \tan\psi\, z_{n+1},$$

and

$$(y_n + a) = \tan\psi\, z_n.$$

Eliminating y_n and y_{n+1} from these three equations, z_{n+1} can be related to z_n as

$$z_{n+1} = \frac{2a + (\tan\phi_n - \tan\psi)z_n}{(\tan\phi_n + \tan\psi)}$$

$$= \frac{2a\cos(\phi + 2n\psi)\cdot\cos\psi + z_n\cdot\sin(\phi + \overline{2n-1}\psi)}{\sin(\phi + \overline{2n+1}\psi)}. \tag{2}$$

When (y_n,z_n) is on S_1 and (y_{n+1},z_{n+}) on S_2, the equation relating the z coordinates of these two points is identical to Eq. (2) because of the symmetry of the structure about the z axis. The difference Eq. (2) is thus valid regardless of whether (y_n,z_n) is on S_1 or on S_2, and its solution then gives

$$z_{n+1} = \frac{2a\cos\psi}{\sin(\phi + \overline{2n+1}\psi)}\cdot\sum_{i=1}^{n}\cos(\phi + 2i\psi) + z_1\frac{\sin(\phi+\psi)}{\sin(\phi + \overline{2n+1}\psi)}.$$

Finally, using a standard formula[4] for the summation,

$$z_{n+1} = \frac{2a\cos\psi}{\sin(\phi + \overline{2n+1}\psi)}\cdot\frac{\cos(\phi + \overline{n+1}\psi)\cdot\sin(n\psi)}{\sin\psi} + z_1\cdot\frac{\sin(\phi+\psi)}{\sin(\phi + \overline{2n+1}\psi)}. \tag{3}$$

The maximum distance traveled by a ray in $n_{\max}$ reflections,

$$z_{n_{\max}+1},$$

can be obtained from Eq. (3). We denote the fraction

$$\frac{\sin(\phi + \overline{2n_{\max}+1}\psi)}{\sin(\phi+\psi)} \text{ by } T. \tag{4}$$

Since

$$2\cos(\phi + \overline{n+1}\psi)\cdot\sin(n\psi) + \sin(\phi+\psi) = \sin(\phi + \overline{2n+1}\psi),$$

we get

$$z_{n_{\max}+1} = \frac{a}{\tan\psi}(1 - 1/T) + z_1/T, \tag{5}$$

where T has the same meaning as in Eq. (4).

In turn z_1 can be expressed as a function of y_0 and the condition for

$$z_{n_{\max}+1} \ge \text{the length of the coupling}$$

tested for different rays. At this point, two cases of incidence angle have to be considered separately.

A. Case 1: $\phi \le \psi$

In this case, a ray starting from $(y_0,0)$ in the direction ϕ has its first reflection on surface S_1 if

$$y_0 > b - (a - b)\tan\phi/\tan\psi. \tag{6}$$

z_1 can therefore be obtained from the intersection of the ray and S_1 as

$$z_1 = (a - y_0)/(\tan\phi + \tan\psi). \tag{7}$$

From Eqs. (5)–(7), the condition on y_0 for the ray to be transmitted by the coupling, i.e.,

$$z_{n_{\max}+1} \ge \text{length of the coupling} = (a - b)/\tan\psi$$

is

$$b(1 + R) - aR < y_0 \le bT(1 + R) - aR, \tag{8}$$

where $R = \tan\phi/\tan\psi$.

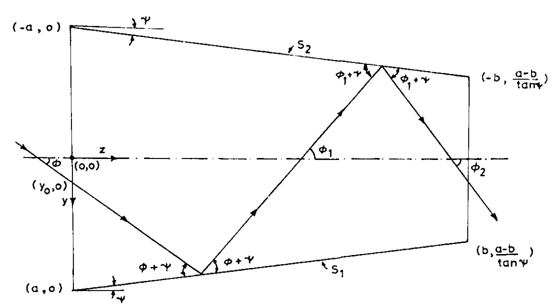

Fig. 2. Coupling projection along the dimension c, the coordinate frame and the progress of a ray along the coupling.

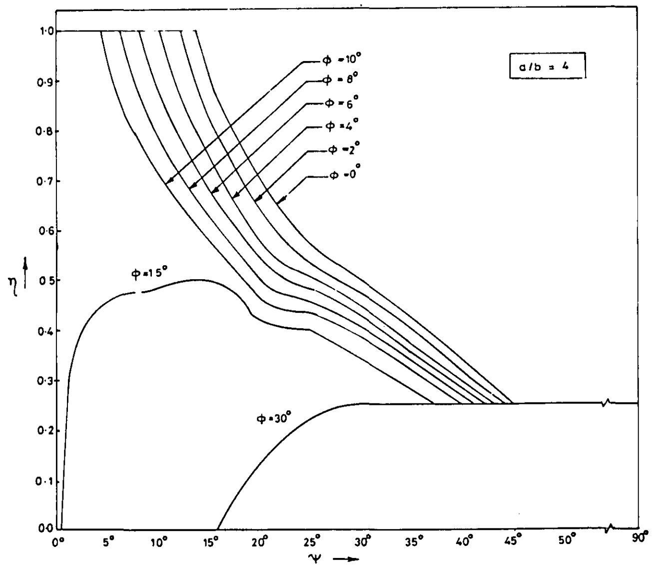

Fig. 3. Efficiency of a coupling as a function of the semicoupling angle ψ for various values of incident radiation angles ϕ and a fixed a/b ratio of 4.

The condition on y_0 such that the first reflection of the ray is on S_2 and

$$z_{n_{\max}+1} \geq (a-b)/\tan\phi$$

can be obtained from Eq. (8) merely by replacing ϕ by $-\phi$ and y_0 by $-y_0$ because of the symmetry of the structure about the z axis as

$$b(1-R) + aR < -y_0 \leq bT(1-R) + aR. \tag{9}$$

Finally, rays that do not have first reflections either on S_1 or on S_2 obviously go through the coupling. For these rays,

$$-b(1-R) - aR \leq y_0 \leq b(1+R) - aR. \tag{10}$$

The sum of the intervals of y_0 defined by Eqs. (8)–(10) for which the ray goes through the coupling gives the efficiency of the coupling in this case as

$$\eta_{\phi\leq\psi} = \frac{1}{2a}[\max\{0, \min\{bT(1+R) - aR, a\} - \max\{-a, b(1+R) - aR\}\} + \max\{0, \min\{bT(1-R) + aR, a\} - \max\{-a, b(1-R) + aR\}\} + 2b]. \tag{11}$$

The min and max expressions in this equation arise because $-a \leq y_0 \leq a$ and to take care of the nonnegativity of each interval. It is shown in the next section that $T \geq 1$ [Eq. (16)]. Also since $\phi \leq \psi$, $R \leq 1$. This simplifies Eq. (11) to

$$\eta_{\phi\leq\psi} = \min\{1, (b/a)T\}. \tag{12}$$

B. Case 2: $\phi > \psi$

In this case, as before, if $a \geq y_0 > b - (a-b)\tan\phi/\tan\psi$, first reflection of the incoming ray is on wall S_1, and the range of y_0 for which the ray is transmitted by the coupling is given by Eq. (8). When $-a \leq y_0 \leq b - (a-b)\tan\phi/\tan\phi$, the rays go through the coupling without any reflection. The efficiency in this case is therefore obtained as

$$\eta_{\phi>\psi} = \frac{1}{2a}(\max\{0, \min\{bT(1+R) - aR, a\} - \max\{-a, b(1+R) - aR\}\} + \max\{0, \min\{a, b(1+R) - aR\} + a\}).$$

Since in this case, $T \geq 1$ and $R > 1$, this expression can be simplified to

$$\eta_{\phi>\psi} = \max\left\{0, \frac{1+R}{2}\min\left\{1, \frac{b}{a}T\right\} - \frac{R-1}{2}\right\}, \tag{13}$$

where $R = \tan\phi/\tan\psi$. Efficiency of a coupling expressed by Eqs. (12) and (13) as a function of its angle ψ is plotted in Fig. 1 for various incident radiation angles ϕ and a/b ratio equal to four.

IV. Design of the Coupling for Maximum Efficiency

It can be seen from Eqs. (12) and (13) that regardless of the relative magnitudes of ϕ and ψ, the necessary and sufficient condition for $\eta = 1$ is

$$bT \geq a. \tag{14}$$

An ideal coupling, i.e., a coupling that can transfer all the incident radiation from the larger light pipe into the smaller light pipe can be designed if condition (14) can be satisfied for some ψ. Function T thus plays an important part in the design of a high efficiency coupling. Some properties of this function are now investigated. For $\psi > \pi/4 - \phi/2$, $n_{\max} = 0$ and $T = 1$. As ψ is reduced $n_{\max}$ increases. It can be proved that for $n_{\max} \geq 1$, $T \geq 1$ as follows:

From Eq. (1), $n_{\max} \leq (\pi - 2\phi)/4\psi < n_{\max} + 1$. Therefore $\pi/2 - \psi < \phi + \overline{2n_{\max}+1}\psi \leq \pi/2 + \psi$. Moreover, $\pi/2 + \psi > \phi + \psi$. We then get from the monotonically increasing nature of sine function in the first quadrant and its symmetry about the value $\pi/2$,

$$\sin(\phi + \overline{2n_{\max}+1}\psi) > \sin(\phi+\psi). \tag{15}$$

Therefore,

$$T \geq 1 \text{ for all } \psi. \tag{16}$$

We can further show that T is monotonically decreasing with ψ.

$$dT/d\psi = [\overline{2n_{\max}+1}\cos(\phi + \overline{2n_{\max}+1}\psi)\sin(\phi+\psi) - \cos(\phi+\psi)\cdot\sin(\phi + \overline{2n_{\max}+1}\psi)]/\sin^2(\phi+\psi). \tag{17}$$

If $(\phi + \overline{2n_{\max}+1}\psi) \geq \pi/2$, $dT/d\psi < 0$, and when $(\phi + \overline{2n_{\max}+1}\psi) < \pi/2$, the sign of $dT/d\psi$ is the same as that of $Q \equiv (2n_{\max}+1)\tan(\phi+\psi) - \tan(\phi + \overline{2n_{\max}+1}\psi)$. It can be shown that

$$\lim_{\psi\to 0} Q \leq 0 \text{ and } dQ/d\psi < 0.$$

Therefore $Q < 0$ for all values of ψ. The function T is thus monotonically decreasing with ψ.

This nature of T indicates that if $\psi_{\max}$ is a solution of $bT = a$, any $\psi \leq \psi_{\max}$ satisfies Eq. (14) and gives a coupling with $\eta = 1$. The values of $\psi_{\max}$ for various incidence angles ϕ are plotted in Fig. 4 as a function

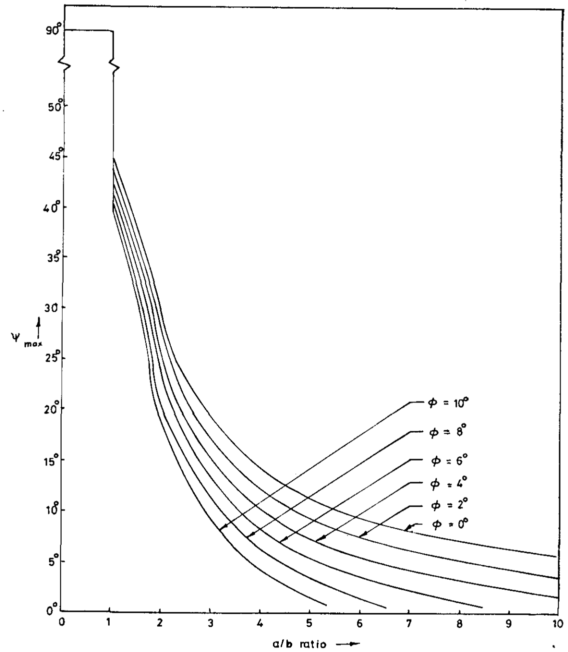

Fig. 4. Maximum semicoupling angle $\psi_{\max}$ for perfect coupling with given a/b ratio and the incident radiation angle ϕ.

of the pipe dimension ratios. For a given a/b ratio and the angle ϕ, these curves can be used to choose a coupling angle ψ [and thereby a coupling length $(a - b)/\tan\psi$] such that efficiency of the coupling is unity.

The monotonically decreasing nature of T also indicates that it is maximum when $\psi = 0$. This value, denoted by $T_{\max}$, is calculated as follows:

$$T_{\max} = \lim_{\psi \to 0} \frac{\sin(\phi + \overline{2n_{\max} + 1}\psi)}{\sin(\phi + \psi)}. \tag{18}$$

$n_{\max}$ is a discontinuous function of ψ. Let $x = (\pi - 2\phi)/4\psi$, a continuous function. Then $|x - n_{\max}| < 1$, and because of this bound,

$$\lim_{\psi \to 0} x\psi = \lim_{\psi \to 0} n_{\max}\psi.$$

The function $2n_{\max}\psi$ in Eq. (18) can therefore be replaced by the function $2x\psi$. The limit can be calculated after this to give

$$T_{\max} = 1/\sin\phi. \tag{19}$$

Combining Eqs. (14) and (19), we can say that if the pipe dimension ratio a/b and the incidence radiation angle ϕ are such that

$$a \sin\phi \geq b, \tag{20}$$

no coupling, however long it may be, can be designed such that its efficiency is unity.

V. Conclusions

The efficiency of a coupling between two rectangular light pipes with specularly reflecting walls is calculated in this paper. The input end of the larger light pipe is assumed to be illuminated uniformly with parallel rays, and this is equivalent to illuminating the entrance plane of the coupling uniformly with parallel rays. This is not exactly true as some rays undergo two extra reflections before they reach the coupling.[2] But if the reflectivity of the walls of the larger light pipe is high, this should not cause a considerable change in the intensities of different rays.

While calculating η, it is assumed that the reflectivity of coupling walls is 1. In practice this may not be true, and the effective coupling efficiency η_e can then be obtained as

$$\eta_e = \eta \cdot \eta_r, \tag{21}$$

where η is the efficiency calculated in this paper, and η_r is the efficiency that depends upon the number of reflections in the coupling and wall reflectivity. η_r can be calculated in case of a given coupling by finding the exact number of reflections from the (tilted as well as parallel) faces of the coupling.

η is dependent largely upon the function T defined by Eq. (4). Equation (14) gives the condition for $\eta = 1$. It is simple to design couplings with $\eta = 1$ for small incidence angles using Fig. 4. For large ϕ on the other hand, it may not be possible to make $\eta = 1$. Equation (20) sets a maximum limit on ϕ for a given pipe dimension ratio a/b or vice versa. For $\phi = 0$, a coupling with $\eta = 1$ can be designed for any pipe dimension ratio.

If a coupling has $\eta = 1$, further increasing its length (i.e., decreasing ψ) does not deteriorate its performance. On the other hand, if $\eta < 1$, increasing the length does not always improve the coupling performance. This is clear from the plots of η against ψ (Fig. 3) for $\psi = 15°$ and $30°$.

The author thanks B. V. Rao of the Indian Institute of Technology, Bombay for his keen interest in this work and helpful discussions.

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