sented in Table II. It is seen that for the same specified error probabilities the ASN for the sequential algorithm is about half that of the Bayes algorithm.

V. Experimental Results

To verify experimentally the performance of the two algorithms, signals from a noise generator (HP 3722A) were used. The generator settings were as follows: mode–infinite, bandwidth–1.5kHz, and RMS value–0.3V. The record length of 50 seconds was digitized (12 bit A/D) and stored on disc using a PDP11/40 minicomputer. The sampling rate used was 1 kHz to ensure independent samples. The two algorithms were then simulated on the PDP11/40 minicomputer in the BASIC language. For both θ and σ^2 the hypotheses were chosen as in (38) and (39).

To generate the different hypotheses, the digitized signal was broken up into five sections $\{S_0, S_1, S_2, S_3, S_4\}$ each 10 000 points long. The appropriate θ was then added to each section, i.e., $\{S_0+\theta_0,\cdots,S_4+\theta_4\}$ was formed, or in the case of σ^2 each section was multiplied by the appropriate factor, i.e., $\{K_0S_0,\cdots,K_4S_4\}$ was formed. The different sections were then applied to the two algorithms. Table II shows the final results for the average number of samples required to make a decision and confirms the analytical results for the ASN.

VI. Conclusions

A sequential multiple hypothesis test for the unknown parameters of a normal distribution has been developed. Since the problem considered is fairly common the results should find applications in many different fields. Analytical expressions are obtained for the average sample number and error probabilities. As far as the authors are aware this is the first *m*ary sequential algorithm for which the performance has been found analytically (at least for $m>3$). It is shown that the ASN for the end hypotheses $\bar{n}_0$ and $\bar{n}_{m-1}$ is approximately equal to the ASN for an optimal binary algorithm. The dependence of ASN for the middle hypotheses $\bar{n}_j$ on the number of hypotheses m is slight. Also, for decreasing error probabilities α and β, $\bar{n}_j$ tends asymptotically (from above) to the value $\bar{n}_0$ (which as mentioned is close to the ASN for the binary case). The error probabilities are the same for the end hypotheses as in the binary case. The principle difference (in terms of performance) between the binary case and *m*ary case is that the error probabilities are approximately doubled for the middle hypotheses of the *m*ary case. In terms of the ASN this is equivalent to increasing $\bar{n}_j$ by about 20 percent. A comparison with a Bayes algorithm shows that the algorithm described here provides a gain (in terms of sample size) of a factor of approximately two. Experimental results obtained for the performance agree closely with the analytical results. The sequential *m*ary algorithm developed here represents a natural generalization of the binary sequential test to an *m*ary one. For $m=2$ this algorithm reduces to the known optimal algorithm.

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References

[1] A. Wald, "Sequential tests of statistical hypotheses," *Ann. Math. Stat.*, vol. 16, pp. 117–186, 1945.
[2] A. Wald, and J. Wolfowitz, "Optimum nature of the sequential probability ratio test," *Ann. Math. Stat.* vol. 19, pp. 326–339, 1948.
[3] P. Armitage, "Some sequential tests of student's hypothesis," *Ann. Math. Stat.*, vol. 2, pp. 250–263, 1947.
[4] M. Sobel and A. Wald, "A sequential decision procedure for choosing one of three hypotheses concerning the unknown mean of a normal distribution," *Ann. Math. Stat.*, vol. 20, pp. 502–522, 1949.
[5] R. C. Reed, "A sequential multi-decision procedure," in *Proc. Symp. on Decision Theory and Appl. Electron. Equipment Develop.*, USAF Develop. Center, Rome, NY, Apr. 1960.
[6] K. Fu, *Sequential Methods in Pattern Recognition and Machine Learning.* New York: Academic, 1968.
[7] T. S. Edrington and D. P. Petersen, "On the performance of some sequential multiple-decision procedures." *IEEE Trans. Aerospace Elect. Sys.*, vol. AES-7, pp. 906–913, Sept. 1971.
[8] S. Benedetto and E. Biglieri, "Sequential decision theory with a map strategy," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 565–569, July 1973.
[9] J. Koplowitz and R. A. Roberts, "Sequential estimation with a finite statistic." *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 631–635, Sept. 1973.
[10] A. Wald, *Sequential Analysis.* New York: Wiley, 1947.
[11] J. J. Stiffler, *Theory of Synchronous Communications.* Englewood Cliffs, NJ: Prentice-Hall, 1971.

Interpolation of Data from Redundancies in Fourier Transforms over Finite Abelian Groups

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***Abstract*—It is shown that if the Fourier transform over an Abelian group G of a sequence is zero except over a set of coset representatives of a subgroup $H'\subset G$, then the sequence can be easily reconstructed from its values over a subgroup $H\subset G$ related to H'.**

I. Introduction

Given a sequence x of $N=n_1n_2\cdots n_t$ components and a finite Abelian group $G\simeq C_{n_1}\times C_{n_2}\times\cdots\times C_{n_r}$, C_{n_t} being a cyclic group of order n_t and generator a_t, one can relabel the sequence component $x(i)$ as x_g, where i $(0\leqslant i\leqslant N-1)$ and g $(\in G)$ are related uniquely by

$$g=i_1a_1+i_2a_2+\cdots+i_ra_r,$$
$$i=i_1n_2n_3\cdots n_r+i_2n_3n_4\cdots n_r+\cdots+i_{r-1}n_r+i_r,$$
$$0\leqslant i_t\leqslant n_t-1,\quad 1\leqslant t\leqslant r.$$

The Fourier transform with respect to G of $x_g, g\in G$, is defined as a sequence X_h, $h\in G$, given by

$$X_h=\sum_{g\in G}\phi_h(g)x_g,\qquad h\in G,$$

where $\phi_h(g)$ is the image of g under the homomorphism ϕ_h from G into the multiplicative group of nonzero complex numbers. Note that the isomorphism of the multiplicative group of such homomorphisms with G allows one to label them with the group elements. The $N\times N$ transform kernel matrix M given by $M(h,g)=\phi_h(g)$, $g,h\in G$, is in fact the character table of G. The inverse transform is given by

$$x_g=\frac{1}{|G|}\sum_{h\in G}\phi_g^*(h)X_h,\qquad g\in G,$$

where the asterisk denotes the complex conjugate. In this correspondence, the transforms of sequences are uniformly denoted by corresponding capital letters.

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Many of the transforms found useful in digital signal processing are Fourier transforms over finite Abelian groups. If $G=C_N$, the resultant transform is a discrete Fourier transform (DFT); for $G=C_2\times C_2\times\cdots\times C_2$, it is a Hadamard transform (HT); and for $G=C_p\times C_p\times\cdots\times C_p$, it is the reordered Chrestenton transform. In general, the transform with respect to $G\simeq C_{n_1}\times C_{n_2}\times\cdots\times C_{n_r}$ has as a kernel a Kronecker product of the Fourier matrices of orders $n_1,n_2,\cdots,n_r$.

Various properties of Fourier transforms with respect to G were discussed earlier by Nicholson [1], Apple and Wintz [2] Karpovsky [3], and Kanetkar and Wagh [4], among others. In particular, a convolution of y_g,z_g, $g\in G$, is a sequence w_h, $h\in G$, defined by

$$w_h=\sum_{g\in G} y_g z_{h-g}, \qquad h\in G,$$

and denoted by $w=y\circledast z$. It is known that if $w=y\circledast z$, then

$$W_g=Y_g\cdot Z_g, \qquad g\in G.$$

In this correspondence we study a discrete analog of the reconstruction of a uniformly sampled bandlimited signal through a procedure which is well-known in the continuous signal domain. In the case of a continuous signal with frequencies between $-B$ and $+B$, it is sufficient to sample it uniformly at the rate of $2B$ samples per unit time. The original signal can be reconstructed from the sampled version by placing the sinc functions of appropriate parameters (which depend on the value of B) and amplitudes (which depend on the sample value) at the sample points and then summing them. We show correspondingly that if the Fourier transform with respect to G of a finite sequence is restricted to a set of coset representatives, then it is possible to obtain the entire sequence from a few of its components in a similar fashion.

II. Main Result

Given a subgroup $H\subset G$, the set

$$H'=\{h'\in G\,|\,\phi_{h'}(h)=1, \text{ all } h\in H\}$$

is a subgroup of G and $H'\simeq G/H$. We denote the set of coset representatives of H' in G by B.

Lemma 1: $[\phi_h^*(g)]$, $g\in B$, $h\in H$, is the character table of H.

Proof: It is sufficient to prove that $A=[\phi_g(h)]$, $g\in B$, $h\in H$, is the character table of H. Obviously each row of A is a character of H. Further, no two rows of A are identical because if

$$\phi_{g_1}(h)=\phi_{g_2}(h), \qquad \text{all } h\in H,$$

then

$$\phi_{g_1-g_2}(h)=1, \qquad \text{all } h\in H.$$

This implies $g_1-g_2\in H'$, hence $g_1=g_2$ because $g_1,g_2\in B$, the set of coset representatives of H'. Thus the $|H|$ rows of A are all the distinct characters of H and A is the character table of H. Q.E.D.

We now show how to compute a sequence z_g, $g\in G$, which vanishes over the nonzero elements of H, i.e., $z_h=0$ if $h\in H$ and $h\neq 0$.

Lemma 2: Let

$$Z_g=|G|/|H|, \qquad \text{if } g\in B,$$
$$=0, \qquad \text{otherwise},$$

then the inverse transform satisfies

$$z_h=1, \qquad \text{if } h=0,$$
$$=0, \qquad \text{if } 0\neq h\in H.$$

Proof:

$$z_h=\frac{1}{|G|}\sum_{g\in G}\phi_h^*(g)Z_g$$
$$=\frac{1}{|H|}\sum_{g\in B}\phi_h^*(g).$$

The result then follows from Lemma 1. Q.E.D.

Lemma 3: If r_g, $g\in G$, and its transform R_g, $g\in G$, satisfy

$$r_g=0, \qquad \text{if } g\in H,$$

and

$$R_g=0, \qquad \text{if } g\notin B,$$

then

$$r_g=0, \qquad \text{for } g\in G.$$

Proof: For $h\in H$,

$$r_h=\frac{1}{|G|}\sum_{g\in G}\phi_h^*(g)R_g$$
$$=\frac{1}{|G|}\sum_{g\in B}\phi_h^*(g)R_g.$$

Since the matrix $[\phi_h^*(g)]$, $g\in B$, $h\in H$, is invertible (from Lemma 1) and the vector on the left side is zero, $R_g=0$, for $g\in B$. This means $R_g=0$, for $g\in G$, from which the required result follows. Q.E.D.

Theorem 1: If x_g is known over H and $X_g=0$, $g\notin B$, then x_g can be constructed fully as $y\circledast z$, where z is the sequence in Lemma 2 and

$$y_g=x_g, \qquad g\in H,$$
$$=0, \qquad \text{otherwise}.$$

Proof: Let $\tilde{x}=y\circledast z$. Then

$$\tilde{x}_h=\sum_{g\in G} y_g z_{h-g}=\sum_{g\in H} x_g z_{h-g}.$$

But when $h\in H$, $h-g\in H$, and therefore from Lemma 2, z_{h-g} vanishes except for $h-g=0$. Thus $\tilde{x}_h=x_h$, if $h\in H$, and $\tilde{X}_g=Y_g\cdot Z_g=0$, $g\notin B$. The result $\tilde{x}_g=x_g$, for $g\in G$, now follows directly by applying Lemma 3 to the sequence $r_g=\tilde{x}_g-x_g$, $g\in G$. Q.E.D.

From the theorem, the sequence x satisfying $X_g=0$, $g\notin B$, can be generated from its values over H by the following method.

i) Compute the sequence z as in Lemma 2.
ii) x is obtained as a weighed sum of $|H|$ shuffled versions of z. Corresponding to each $h\in H$, the shuffling is $z_g\to z_{h-g}$ and the weighing factor is x_h.

For example, in the case of a DFT of length 16 (which is a Fourier transform with respect to $C_{16}=\langle a\rangle$), if the only nonzero components of X are $X(0)$, $X(1)$, $X(14)$, and $X(15)$ (corresponding to $B=\{0,a,14a,15a\}$), the complete sequence x can be generated from $x(0)$, $x(4)$, $x(8)$, and $x(12)$ (corresponding to $H=\langle 4a\rangle=\{0,4a,8a,12a\}$) as follows.

i) For the given B, $z(i)=(1+w^i+w^{14i}+w^{15i})/4$, where w is a sixteenth primitive root of unity.
ii) $$x(i)=x(0)\cdot z(-i)+x(4)\cdot z(4-i)+x(8)\cdot z(8-i)+x(12)\cdot z(12-i),$$
where the indices are evaluated modulo 16.

Note that for any given proper subgroup $H\subset G$, B is not unique. This increases the probability of the transform being restricted to some B, thus improving the applicability of the technique under discussion. For example, in the above case, there are 4^4 different sets B defined by $B=\{i_0a, i_1a, i_2a, i_3a\}$, where $i_t \equiv t \pmod 4$, $t=0,1,2,3$.

The reconstruction procedure described here is analogous to the construction of a bandlimited signal from its samples. The restriction of the transform energy in our case to the set B corresponds to the bandlimitedness in the continuous domain. The z sequence which depends only on the set B corresponds to the sinc function used in the continuous domain. The sinc also depends only on the bandwidth. Finally, our reconstructed sequence is a weighed sum of shuffled z sequences. Analogously, in the continuous domain, the reconstructed signal is a weighed sum of shifted sinc functions. The weighing constants in both cases are the known signal samples.

References

[1] P. J. Nicholson, "Algebraic theory of finite Fourier transform," *J. Comput. Syst. Sci.*, vol. 5, pp. 524–547, 1971.

[2] G. Apple and P. Wintz, "Calculation of Fourier transform on finite Abelian groups," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 233–234, 1970.

[3] M. G. Karpovsky, *Finite Orthogonal Series in the Design of Digital Devices (Analysis, Synthesis and Optimization).* New York: Wiley, 1976.

[4] S. V. Kanetkar and M. D. Wagh, "Group character tables in discrete transform theory," *J. Comput. Syst. Sci.*, vol. 19, pp. 1–11, 1979.