Butterfly Automorphisms and Edge Faults

Meghanad D. Wagh and Khadidja Bendjilali
Department of Electrical and Computer Engineering
Lehigh University, Bethlehem, PA 18015, USA
mdw0@Lehigh.Edu, khb5@Lehigh.Edu

***Abstract*—This paper obtains all the automorphisms of a wrapped butterfly network of degree n using an algebraic model. It also investigates the translation of butterfly edges by automorphisms. It proposes a new strategy for algorithm mappings on an architecture with faulty edges. This strategy essentially consists of finding an automorphism that would map the faulty edges to the free edges in the graph. Having a set of $n2^{n+1}$ well defined simple automorphisms which translate graph edges deterministically, makes this a very powerful technique for dealing with edge faults. We illustrate the technique by mapping Hamilton cycle on the butterfly under various edge fault scenarios.**

I. Introduction

Last few decades have seen rapid development of the semiconductor technology resulting in faster computing devices. However, during the same period, the need for computing resources has increased much faster. As a result, parallel machines with many processors working simultaneously on the same problem have become a necessity.

Unfortunately, the speed of data transfers between cooperating processors has not kept pace with the increase in the computing speed. Consequently, the performance of distributed memory parallel machines is often governed by the underlying interconnection networks. The choice of the interconnection network also affects other key characteristics of the system such as the ease of algorithm development, reliability, scalability and complexity of physical layout. These networks can be modeled as graphs whose nodes represent processors, and edges, the communication paths between them. Hypercubes, butterflies and meshes are some of the popular graphs on which many of the existing parallel machines are based [1].

The *wrap-around butterfly network* represents a good tradeoff between the cost and the performance of a parallel machine. It has a large number of processors, fixed node degree, low diameter, symmetry, and ability to support a variety of parallel algorithms. Cube Connected Cycles is a sub-graph of B_n [2]. Other extensions of B_n are also available [3, 4]. B_n supports many parallel algorithms efficiently [1, 5–11].

Let Z_n denote the group of integers $\{0, 1, \ldots, n-1\}$ under the operation of addition modulo n and Z_2^n, the group of binary vectors of length n under the operation of modulo 2 addition. Then the wrapped butterfly graph B_n, $n \geq 3$, is defined to have $n2^n$ nodes, each labeled with a pair (m, V) where $m \in Z_n$ and $V \in Z_2^n$. A node (m, V) is connected to four distinct nodes: $(m+1, V)$, $(m+1, V \oplus 2^m)$, $(m-1, V)$ and $(m-1, V \oplus 2^{m-1})$. Note that the third and the fourth edges are inverses of the first and the second edges respectively. Thus the edges of a wrapped butterfly are bidirectional. The first index m of the node (m, V) is often called its column and the second index, V, its row.

With the advances in the VLSI technology, it is now possible to build parallel machines with a large number of processors. However, larger the machine, higher is the probability that one or more of its processors or links will develop a fault. Thus, for the underlying networks of these large machines, mappings of algorithms on faulty graphs becomes an important design issue.

Previous results about mappings on faulty butterflies include one by Vadapalli and Srimani who have shown that in B_n, there exists a cycle of length at least $n2^n - 2$ with one faulty node and $n2^n - 4$ with two faulty nodes [12]. Later, Tsai et al., improved this to show that for odd n, cycle length $n2^n - 2$ is possible with two faulty nodes [13]. They also proved that in the presence of one faulty node and one faulty edge, there exists a cycle of length $n2^n - 2$ when n is even and $n2^n - 1$ when n is odd. Hwang and Chen have shown that the maximal cycle of length $n2^n$ can be embedded in a faulty butterfly even with two edge faults [14]. However, these studies have used the binary representation of the butterfly resulting in rather complex mappings.

This paper proposes a new approach to mappings on faulty butterflies using an algebraic model first given in [11]. We show that with this model, it is rather simple to obtain all the automorphisms of the butterfly. Automorphisms can be used to translate an algorithm mapping to one that avoids node faults. For example, an algorithm mapping can avoid a faulty node N_{faulty} by using a free node N_{free} (assuming one exists) and an automorphism $\phi(\cdot)$ of the interconnection graph such that $\phi(N_{free}) = N_{faulty}$. By remapping tasks on each node N to node $\phi(N)$, one can run the algorithm entirely on fault free nodes. Automorphisms have also used to obtain better VLSI layouts of butterfly networks [15, 16].

We explore the edge transformations in butterfly networks due to automorphisms. This allows one to map algorithms onto butterfly machines with edge faults. As an example, we show that a butterfly B_n supports a Hamilton cycle even when it has up to 2^n faulty edges of the same type (to be defined later) in each column except one. As a corollary, one can show that B_n is Hamiltonian with up to $n-1$ random edge faults distributed one per column. Our procedure allows one to map the Hamilton cycle on to the faulty butterfly easily and directly.

978-0-7695-4120-4/10 $26.00 © 2010 IEEE
DOI 10.1109/ISPDC.2010.11

The simplicity of the automorphism and the resultant edge mappings show promise of wide applicability of this technique to a variety of applications.

II. An algebraic model of the butterfly

Binary representation has been widely used to model many common interconnection networks including the butterfly. However, binary models are difficult to analyze and complex to use. In this paper we will use an algebraic model using direct product of finite fields and cyclic groups, first given in [11]. The simplicity of the model and access to powerful algebraic techniques allows us to explore the automorphisms of the butterfly with relative ease.

In the butterfly model of [11], nodes of B_n are labeled with pairs (m, X), $m \in C_n$, $X \in GF(2^n)$, where C_n is the cyclic group of integers 0 through $n-1$ under the operation of addition modulo n and $GF(2^n)$ is the finite field of 2^n elements. Let α denote the primitive element of $GF(2^n)$ and $\langle \beta_{n-1}, \beta_{n-2}, \ldots, \beta_0 \rangle$, its dual basis. The node connectivity of graph B_n can then be described through an algebraic relationship. In particular, a vertex (m, X) of B_n is connected to the vertices $(m+1, \alpha X)$, $(m+1, \alpha X + \beta_{n-1})$, $(m-1, \alpha^{-1}X)$ and $(m-1, \alpha^{-1}X + \beta_0)$. For convenience, We refer to these four edges as f, g, f^{-1} and g^{-1} respectively. It is easy to verify that if edge f goes from node N_1 to N_2, then the edge that goes from N_2 to N_1 is f^{-1}. The same observation is also true for g and g^{-1}. The simplicity of this model should be apparent from the fact that the two components of the destination of (m, X) are independent. On the other hand, in binary representation, the destination of (m, V) is $(m+1, V \oplus 2^m)$, where, as one can see, the second coordinate is a function of both m and V, the two coordinates of the source. For the proof and examples of the algebraic model, reader is referred to [11]. For later reference, we provide the relationships between the elements of $GF(2^4)$ used in the definition of B_4 in Table I.

TABLE I
Structure of $GF(2^4)$.

Primitive Polynomial: x^4+x+1 Elements and their Relationships:	
0	$\alpha^7 = \alpha^3 + \alpha + 1$
1	$\alpha^8 = \alpha^2 + 1$
α	$\alpha^9 = \alpha^3 + \alpha$
α^2	$\alpha^{10} = \alpha^2 + \alpha + 1$
α^3	$\alpha^{11} = \alpha^3 + \alpha^2 + \alpha$
$\alpha^4 = \alpha + 1$	$\alpha^{12} = \alpha^3 + \alpha^2 + \alpha + 1$
$\alpha^5 = \alpha^2 + \alpha$	$\alpha^{13} = \alpha^3 + \alpha^2 + 1$
$\alpha^6 = \alpha^3 + \alpha^2$	$\alpha^{14} = \alpha^3 + 1$
Dual Base $\langle \beta_3, \beta_2, \beta_1, \beta_0 \rangle = \langle 1, \alpha, \alpha^2, \alpha^{14} \rangle$.	

For the purpose of this paper, one need not worry about the dual basis elements, except that they are constants satisfying the properties given in the following Lemma.

Lemma 1: Let $\langle \beta_{n-1}, \beta_{n-2}, \ldots, \beta_0 \rangle$ denote the dual base of GF(2^n). Then

$$\beta_i = \begin{cases} \alpha\beta_0 & \text{if} \quad i = n-1 \\ \alpha\beta_{i+1} + p_{i+1}\beta_{n-1} & \text{if} \quad i = 0, 1, \ldots, n-2, \end{cases}$$

where α is the primitive element of the field and p_i is the coefficient of x^i in the primitive polynomial used to generate the field.

Proof. Omitted for brevity. ∎

III. Automorphisms of the butterfly network

Wagh and Guzide have previously shown that the algebraic model allows efficient mappings of cycles of all (possible) lengths and trees of largest sizes on the butterfly [11]. We now explore the automorphisms of butterfly in the same setting.

We first give the following lemma which relates the edges in a column to edges in any other column.

Lemma 2: **(connectivity)** Let $K_m, K_{m+1} \in GF(2^n)$ be related as $K_{m+1} = \alpha K_m$ or $K_{m+1} = \alpha K_m + \beta_{n-1}$. For any $X, Y \in GF(2^n)$ and $t \in Z_n$, if nodes (m, X) and $(m+1, Y)$ are connected in B_n, then so are the nodes $(m+t, X+K_m)$ and $(m+1+t, Y+K_{m+1})$.

Proof. The presence of the edge $(m+t, X+K_m) \rightarrow (m+1+t, Y+K_{m+1})$ can be proved by showing that $Y + K_{m+1} = \alpha(X+K_m) + c\beta_{n-1}$ for some $c \in \{0,1\}$. Since $(m, X) \rightarrow (m+1, Y)$, the connectivity of B_n gives $Y = \alpha X + c'\beta_{n-1}$ for $c' \in \{0,1\}$. Further, the given constants K_m and K_{m+1} are related as $K_{m+1} = \alpha K_m + c''\beta_{n-1}$, where $c'' \in \{0,1\}$. Therefore $Y + K_{m+1} = \alpha(X+K_m) + (c'+c'')\beta_{n-1}$. ∎

The connectivity specified by Lemma 2 can be used to obtain the automorphisms of the butterfly network as shown in Theorem 1.

Theorem 1: If constants $K_0, K_1, \ldots, K_{n-1} \in GF(2^n)$ satisfy

$$K_i = \begin{cases} \alpha K_{i-1} \ \text{ or } \ \alpha K_{i-1} + \beta_{n-1}, & \text{if } 0 < i \leq n-1, \\ \alpha K_{n-1} \ \text{ or } \ \alpha K_{n-1} + \beta_{n-1} & \text{if } i = 0, \end{cases}$$

then function $\phi(\cdot) : B_n \rightarrow B_n$ defined as

$$\phi((m,\ X)) = (m+t,\ X+K_m) \tag{1}$$

for any $t \in Z_n$, is an automorphism of B_n, i.e., it maps nodes of B_n to nodes and edges to edges.

Proof. The fact that $\phi(\cdot)$ maps edges to edges is clear from Lemma 2. To prove that it is an automorphism we only have to show that it is a one-to-one and onto mapping.

Let $\phi(m, X) = \phi(m', X')$, then from the definition of $\phi(\cdot)$,

$$(m+t, X+K_m) = (m'+t, X'+K_{m'}).$$

From the first components of the two pairs, $m = m'$. From the second components, $X + K_m = X' + K_m$ which implies that $X = X'$. Thus two distinct nodes cannot have the same image under $\phi(\cdot)$, i.e., $\phi(\cdot)$ is one-to-one.

Now consider any node $(m', Y) \in B_n$. It is easy to see that this node is the image of $(m' - t, Y + K_{m'-t})$. Therefore $\phi(\cdot)$ is onto. $\blacksquare$

Note that constant t merely translates edges in one column to a column t away. As Theorem 1 shows, this t and constant elements $K_i \in GF(2^n)$, $0 \le i < n$ fully define the automorphism $\phi(\cdot)$. We will henceforth refer to t as the *column offset* and K_is as the *automorphism offsets*,

One can see the simplicity of the automorphism $\phi(\cdot)$ defined in (1). Every node in the network is applied the same column offset and every node in the same column is applied the same automorphism offset. Further, the offsets of the two coordinates of a node label are *independent*. This makes use of such an automorphism especially attractive.

Theorem 1 allows one to design such an automorphism under various conditions. For example, suppose one wants an automorphism such that for a given pair of nodes $N_1 = (a,\ U), N_2 = (b,\ V) \in B_n$, the automorphism maps N_1 to N_2, i.e.,

$$\phi(N_1) = N_2. \tag{2}$$

(If we can do this for an arbitrary pair of nodes, it would imply that B_n is a symmetric network.) Such a mapping can be obtained by choosing a column offset t and automorphism offsets $K_0, K_1, \ldots, K_{n-1} \in GF(2^n)$ satisfying condition in Theorem 1) and then defining ϕ as in (1). Note that the relations between K_is provide certain flexibility in the choice of the constants. We exploit this flexibility to ensure that (2) is satisfied.

Let us rewrite the relations between K_is as

$$K_i = \alpha K_{(i-1) \mod n} + c_i \beta_{n-1}, \qquad 0 \le i \le n-1, \tag{3}$$

where each c_i is either 0 or 1. One can use (3) repeatedly to express any individual automorphism offset as

$$\begin{aligned} K_a &= \alpha K_{(a-1) \mod n} + c_a \beta_{n-1} \\ &= \alpha^2 K_{(a-2) \mod n} + (c_{(a-1) \mod n}\ \alpha + c_a)\beta_{n-1} \\ &= \alpha^3 K_{(a-3) \mod n} + \\ &\quad (c_{(a-2) \mod n}\ \alpha^2 + c_{(a-1) \mod n}\ \alpha + c_a)\beta_{n-1}. \end{aligned}$$

Proceeding in this fashion, one gets

$$K_a = \alpha^n\ K_a + (\sum_{j=0}^{n-1} c_{(a-j) \mod n}\ \alpha^j)\beta_{n-1},$$

or

$$K_a = (1 + \alpha^n)^{-1} (\sum_{j=0}^{n-1} c_{(a-j) \mod n}\ \alpha^j)\beta_{n-1}. \tag{4}$$

Further, if $\phi((m,\ X)) = (m + t,\ X + K_m)$, then to satisfy (2) requires that

$$\begin{aligned} t &= (b - a) \mod n \qquad \text{and} \\ K_a &= U + V. \end{aligned} \tag{5}$$

TABLE II
AUTOMORPHISM $\phi(\cdot) : B_4 \to B_4$ SUCH THAT $\phi(3, \alpha^{14}) = (1, \alpha^2)$.

(m, X)	$\phi(m, X)$
$(0,0)$	$(2,\alpha^3)$
$(0,1)$	$(2,\alpha^{14})$
$(0,\alpha)$	$(2,\alpha^9)$
$(0,\alpha^2)$	$(2,\alpha^6)$
$(0,\alpha^3)$	$(2,0)$
$(0,\alpha^4)$	$(2,\alpha^7)$
$(0,\alpha^5)$	$(2,\alpha^{11})$
$(0,\alpha^6)$	$(2,\alpha^2)$
$(0,\alpha^7)$	$(2,\alpha^4)$
$(0,\alpha^8)$	$(2,\alpha^{13})$
$(0,\alpha^9)$	$(2,\alpha)$
$(0,\alpha^{10})$	$(2,\alpha^{12})$
$(0,\alpha^{11})$	$(2,\alpha^5)$
$(0,\alpha^{12})$	$(2,\alpha^{10})$
$(0,\alpha^{13})$	$(2,\alpha^8)$
$(0,\alpha^{14})$	$(2,1)$
$(1,0)$	$(3,\alpha^4)$
$(1,1)$	$(3,\alpha)$
$(1,\alpha)$	$(3,1)$
$(1,\alpha^2)$	$(3,\alpha^{10})$
$(1,\alpha^3)$	$(3,\alpha^7)$
$(1,\alpha^4)$	$(3,0)$
$(1,\alpha^5)$	$(3,\alpha^8)$
$(1,\alpha^6)$	$(3,\alpha^{12})$
$(1,\alpha^7)$	$(3,\alpha^3)$
$(1,\alpha^8)$	$(3,\alpha^5)$
$(1,\alpha^9)$	$(3,\alpha^{14})$
$(1,\alpha^{10})$	$(3,\alpha^2)$
$(1,\alpha^{11})$	$(3,\alpha^{13})$
$(1,\alpha^{12})$	$(3,\alpha^6)$
$(1,\alpha^{13})$	$(3,\alpha^{11})$
$(1,\alpha^{14})$	$(3,\alpha^9)$

(m, X)	$\phi(m, X)$
$(2,0)$	$(0,\alpha^5)$
$(2,1)$	$(0,\alpha^{10})$
$(2,\alpha)$	$(0,\alpha^2)$
$(2,\alpha^2)$	$(0,\alpha)$
$(2,\alpha^3)$	$(0,\alpha^{11})$
$(2,\alpha^4)$	$(0,\alpha^8)$
$(2,\alpha^5)$	$(0,0)$
$(2,\alpha^6)$	$(0,\alpha^9)$
$(2,\alpha^7)$	$(0,\alpha^{13})$
$(2,\alpha^8)$	$(0,\alpha^4)$
$(2,\alpha^9)$	$(0,\alpha^6)$
$(2,\alpha^{10})$	$(0,1)$
$(2,\alpha^{11})$	$(0,\alpha^3)$
$(2,\alpha^{12})$	$(0,\alpha^{14})$
$(2,\alpha^{13})$	$(0,\alpha^7)$
$(2,\alpha^{14})$	$(0,\alpha^{12})$
$(3,0)$	$(1,\alpha^{13})$
$(3,1)$	$(1,\alpha^6)$
$(3,\alpha)$	$(1,\alpha^{12})$
$(3,\alpha^2)$	$(1,\alpha^{14})$
$(3,\alpha^3)$	$(1,\alpha^8)$
$(3,\alpha^4)$	$(1,\alpha^{11})$
$(3,\alpha^5)$	$(1,\alpha^7)$
$(3,\alpha^6)$	$(1,1)$
$(3,\alpha^7)$	$(1,\alpha^5)$
$(3,\alpha^8)$	$(1,\alpha^3)$
$(3,\alpha^9)$	$(1,\alpha^{10})$
$(3,\alpha^{10})$	$(1,\alpha^9)$
$(3,\alpha^{11})$	$(1,\alpha^4)$
$(3,\alpha^{12})$	$(1,\alpha)$
$(3,\alpha^{13})$	$(1,0)$
$(3,\alpha^{14})$	$(1,\alpha^2)$

By combining (4) and (5), one gets

$$(U + V)(\alpha^n + 1)\beta_{n-1}^{-1} = \sum_{j=0}^{n-1} c_{(a-j) \mod n} \alpha^j, \tag{6}$$

One can see that the left hand side of (6) is an element of $GF(2^n)$ and can therefore be uniquely expressed in the normal basis $\langle \alpha^{n-1}, \alpha^{n-2}, \ldots, 1 \rangle$. This gives the unique set of values for c_is. One can then use these values in (3) to obtain the automorphism offsets $K_{(a+1) \mod n}$, $K_{(a+2) \mod n}$, $\ldots$, $K_{(a-1) \mod n}$.

One can illustrate this procedure by computing an automorphism $\phi(\cdot) : B_4 \to B_4$ which maps node $(3, \alpha^{14})$ to node $(1, \alpha^2)$. For this function, the column offset $t = (1 - 3) \mod 4 = 2$ and the automorphism offset $K_3 = \alpha^{14} + \alpha^2 = \alpha^{13}$. (see Table I.) Further,

$$\sum_{j=0}^{n-1} c_{(3-j) \mod n} \alpha^j = K_3 \sigma \beta_3^{-1} = \alpha^3 + 1.$$

Thus one gets $c_0 = 1$, $c_1 = 0$, $c_2 = 0$ and $c_3 = 1$ and consequently, $K_0 = \alpha^3$, $K_1 = \alpha^4$ and $K_2 = \alpha^5$. The resultant automorphism function $\phi(\cdot)$ is given in Table II.

It is easy to verify that the mapping in Table II preserves connectivity.

As is evident from this discussion, all the automorphism offsets for any $\phi(\cdot)$ are related such that choosing any one of them, say, K_0, fixes all the others. On the other hand, distinct K_0 and t values give rise to distinct automorphisms. Thus there are exactly $n2^n$ automorphisms of butterfly B_n when the first index of all the nodes is translated by the same amount.

Because the automorphism offsets play such a central role in defining the automorphism, we now provide some of their basic properties.

Theorem 2: Let $\phi(\cdot), \phi'(\cdot) : B_n \rightarrow B_n$ be any two automorphisms of B_n based on sets of constants $t, K_0, K_1, \ldots K_{n-1}$ and $t', K'_0, K'_1, \ldots K'_{n-1}$. Then,

1) If any $K_m = 0$, then all $K_i = 0$, $0 \leq i < n$.
2) If any $K_m \neq 0$, then all $K_i \neq 0$, $0 \leq i < n$.
3) If any $K_m = K'_m$, then all $K_i = K'_i$, $0 \leq i < n$.
4) If any $K_m \neq K'_m$, then for every i, $0 \leq i < n$, $K_i \neq K'_i$.
5) $\sum_{i=0}^{n-1} K_i$ is either 0 or $(1+\alpha)^{-1}\beta_{n-1}$.

Proof. From (4) one can see that $K_m = 0$ implies that $c_j = 0$, $0 \leq j < n$. Relation (3) then shows that each K_i is zero. On the other hand, if any K_m is nonzero, then so is every other K_i or else, any $K_i = 0$ would invalidate any other nonzero K_m. This proves the first two parts of the corollary.

To prove the third and fourth parts, it is sufficient to note from (3) and (4) that any given K_m uniquely determines all the other K_is. If $K_m = K'_m$, then from (6) we get the same c values in the two cases, which will generate an equal set of K values. Hence, $K_i = K'_i$, for all i.

Finally, the sum of all K_is can be computed as follows. By applying a summation to both sides of (3), one gets

$$\begin{aligned} \sum_{i=0}^{n-1} K_i &= \alpha(\sum_{i=0}^{n-1} K_i) + (\sum_{i=0}^{n-1} c_i)\beta_{n-1} \\ &= (\sum_{i=0}^{n-1} c_i)\beta_{n-1}(1+\alpha)^{-1} \end{aligned} \tag{7}$$

Since $\sum_{i=0}^{n-1} c_i$ in (7) is either 0 or 1, the sum of all K_is is as stated in the Corollary. ∎

We now investigate another automorphism of B_n that reflects the column index of each node.

Theorem 3: For every $X \in GF(2^n)$, $X = \sum_{i=0}^{n-1} x_i\beta_i$, let $X' = \sum_{i=0}^{n-1} x_i\beta_{n-1-i}$. Then the mapping

$$\psi(m, X) = (n-m, X')$$

is an automorphism of B_n.

Proof. It is simple to see that $\psi(\cdot)$ is one-to-one and onto. We only need to prove that it preserves the edge connectivity of B_n. In particular, we demonstrate that since vertex (m, X) is connected to the vertices $(m+1, \alpha X + c\beta_{n-1})$, $c \in \{0,1\}$, $\psi(m, X)$ is also connected to vertices $\psi(m+1, \alpha X + c\beta_{n-1})$. Let $X = \sum_{i=0}^{n-1} x_i\beta_i$. Then using the relationships between the consecutive β_is given in Lemma 1, one gets

$$\begin{aligned} \alpha X + c\beta_{n-1} &= \sum_{i=1}^{n-1}(x_i\beta_{i-1} + x_i p_i \beta_{n-1}) + (c + x_0)\beta_0 \\ &= \sum_{i=0}^{n-2} x_{i+1}\beta_i + (c + \sum_{i=0}^{n-1} p_i x_i)\beta_{n-1}. \end{aligned} \tag{8}$$

Thus

$$\psi(m+1, \alpha X + c\beta_{n-1}) = (n-m-1, Y) \tag{9}$$

where,

$$\begin{aligned} Y &= \sum_{i=0}^{n-2} x_{i+1}\beta_{n-1-i} + (c + \sum_{i=0}^{n-1} p_i x_i)\beta_0 \\ &= \sum_{i=1}^{n-1} x_i\beta_{n-i} + (c + \sum_{i=0}^{n-1} p_i x_i)\beta_0) \end{aligned} \tag{10}$$

Now,

$$\begin{aligned} \alpha Y &= \sum_{i=1}^{n-1} x_i\alpha\beta_{n-i} + (c + \sum_{i=0}^{n-1} p_i x_i)\beta_{n-1}) \\ &= \sum_{i=1}^{n-1}(x_i\beta_{n-1-i} + x_i p_{n-i}\beta_{n-1}) \\ &\quad + (c + \sum_{i=0}^{n-1} p_i x_i)\beta_{n-1}) \\ &= \sum_{i=0}^{n-1} x_i\beta_{n-1-i} + c'\beta_{n-1}, \end{aligned} \tag{11}$$

where $c' \in \{0,1\}$ denotes

$$c' = c + \sum_{i=0}^{n-1}(p_i + p_{n-i})x_i. \tag{12}$$

Note that

$$\begin{aligned} \psi(m, X) &= (n-m, \sum_{i=0}^{n-1} x_i\beta_{n-1-i}) \\ &= (n-m, \alpha Y + c'\beta_{n-1}). \end{aligned} \tag{13}$$

From (9) and (13) it is obvious that vertex $\psi(m, X)$ is connected to vertex $\psi(m+1, \alpha X + c\beta_{n-1})$, $c \in \{0,1\}$. ∎

Unlike the $\phi(\cdot)$ automorphisms investigated earlier, $\psi(\cdot)$ maps some of the butterfly vertices to the same rows (but may be to different columns) as stated by the following theorem.

Theorem 4: In every column of B_n, there are exactly $2^{\lceil n/2 \rceil}$ nodes that do not change their row index under automorphism $\psi(\cdot)$.

Proof. Consider a node $(m, X) \in B_n$, where X is expressed in its dual basis as $X = \sum_{i=0}^{n-1} x_i\beta_i$. As a consequence of Theorem 3, $\psi((m, X)) = (n-m, X)$ if and only if $x_i = x_{n-1-i}$, for $i = 0, 1, .., \lfloor n/2 \rfloor$. Thus the first half of x_is determine the last half. Therefore there are exactly $2^{\lceil n/2 \rceil}$ values of $X \in GF(2^n)$ that will satisfy $\psi((m, X)) = (n-m, X)$. ∎

We end this section with the following theorem enumerating all the automorphisms of B_n.

Theorem 5: B_n has a total of $n2^{n+1}$ automorphisms.

Proof. Note that the product of two automorphisms is also an automorphism. Thus in addition to the $n2^n$ automorphisms defined by Theorem 1, another set of automorphisms can be defined by multiplying each of these $\phi(\cdot)$s by the automorphism $\psi(\cdot)$ in Theorem 3. Since the order of automorphism $\psi(\cdot)$ is 2, these are all the automorphisms of B_n. ∎

IV. Edge Transformations by automorphisms

This section investigates the effect of an automorphism on the butterfly edges. We call edges $(i-1, X) \rightarrow (i, \alpha X)$ and $(i-1, X) \rightarrow (i, \alpha X + \beta_{n-1})$ for all $X \in GF(2^n)$ as the edges in the ith column of B_n.

The automorphism $\phi(\cdot)$ of Theorem 1 affects all the edges in the same column similarly as shown below.

Theorem 6: Let the automorphism offsets be related as:

$$K_i = \alpha K_{(i-1) \mod n} + c_i \beta_{n-1}, \qquad 0 \leq i \leq n-1,$$

(a) If $c_i = 1$, then the automorphism $\phi(\cdot)$ maps all f edges of B_n in column i to g edges and all g edges to f edges.
(b) If $c_i = 0$, then the automorphism $\phi(\cdot)$ maps all f edges of B_n in column i to f edges and all g edges to g edges.

Proof. Consider an f edge between nodes $N_1 = (i-1, X)$ and $N_2 = (i, \alpha X)$ of the sub-graph of B_n. Now, $\phi(N_1) = (i-1, X + K_{i-1})$ and,

$$\begin{aligned} \phi(N_2) &= (i, \alpha X + K_i) \\ &= (i, \alpha X + \alpha K_{i-1} + c_i \beta_{n-1}) \\ &= (i, \alpha(X + K_{i-1}) + \beta_{n-1}) \end{aligned}$$

From this, one can clearly see that the edge between $\phi(N_1)$ and $\phi(N_2)$ is a g edge. The translation of a g edge into an f edge can be similarly proved. ∎

Note that the automorphism $\phi((m,\ X)) = (m+t,\ X+K_m)$ also advances the column number m by quantity t. In this case, $c_m = 1$ has the effect of mapping the f edges of the sub-graph between columns $m-1$ and m to g edges and all g edges to f edges; but these transformed edges now appear in column $m+t$. Similarly the edges in mth column are mapped to edges of the same type in column $m+t$ if $c_m = 0$.

We will show in the next section how Theorem 6 is helpful in avoiding faulty edges in a mapping. However, for completeness, we provide below results (without proofs) relating to the effect of automorphism $\psi(\cdot)$ on edges of the butterfly.

Theorem 7: Automorphism ψ maps f edges from (m, X) to f edges and g edges from (m, X) to g edges if and only if

$$\psi(X) = \alpha \psi(\alpha X)$$

Theorem 8: Edges from exactly half the nodes in every column of the butterfly are mapped to edges of the same type.

Theorem 7 shows that edges starting from nodes in the same row (i.e., nodes (m, X) having the same X) behave similarly, i.e., all of them either map to edges of the same type (f to f, g to g) or map to edges of the other type (f to g, g to f). Further, Theorem 8 shows that there are equal number of rows of each kind.

V. Application of automorphisms to tolerate edge faults

Previously automorphisms have only been used to tolerate node faults. However, Theorems 6 and 7 directly express the effect of an automorphism on the butterfly edges. Consequently, one can now use these automorphisms to tolerate edge faults for many mappings on the butterfly.

The general procedure to obtain a fault free mapping on a faulty butterfly is simple. If some edges used in the mapping are faulty but the edges to which they *can* be mapped by *some* automorphism are free, then applying that automorphism to the mapping will allow it to use only fault-free edges. Note that much of the power of this method is due to the fact that we have $n2^{n+1}$ well-defined and simple automorphisms that map edges in a deterministic fashion. We illustrate this procedure by constructing a Hamilton cycle under various edge fault scenarios.

Theorem 9: If the edges in one of the columns of B_n are fault free and the faults in each of the other columns are limited to only one type of edges, then B_n is Hamiltonian.

Proof. As shown in [11], it is possible to construct a Hamiltonian cycle in B_n by first constructing two cycles using only f edges; one linking all nodes (m, X), $X \neq 0$, and another linking all nodes $(m, 0)$. These cycles are merged into a Hamiltonian cycle by using a pair of g edges in column t: $(t-1, 0) \rightarrow (t, \beta_{n-1})$ and $(t-1, \beta_0)$ and $(t, 0)$. With $0 \leq t < n$, there are n such independent pairs of g edges that may be used to merge the cycles. We will use the g edges in the column of B_n that has no faults. We now show that one can design an automorphism $\phi : B_n \rightarrow B_n$ which will avoid all faults. To construct ϕ, we compute constants c_i, $0 \leq i < n$ such that

$$c_i = \begin{cases} 1 & \text{if there is a fault in } f \text{ edge in column } i \\ 0 & \text{otherwise} \end{cases} \qquad (14)$$

One can then get K_0 by (6) as

$$K_0(\alpha^n + 1)\beta_{n-1}^{-1} = \sum_{j=0}^{n-1} c_{(-j) \ mod \ n} \alpha^j .$$

The other K_i values can then be inferred from (3). Theorem 6 then shows that the Hamilton cycle will use f edges in columns where f edges are fault free and g edges where f edges have faults. Thus the transformed Hamiltonian cycle will not have any faulty edges. ∎

To illustrate Theorem 9, consider a butterfly B_4 shown in Fig. 1 with faults in columns 0 and 1 restricted to f edges and in column 2 to g edges. Edges in column 3 are fault free. Clearly in this case, $c_0 = c_1 = 1$ and $c_2 = c_3 = 0$. This gives from (14), $K_0 = \alpha^{13}$, $K_1 = \alpha^3$, $K_2 = \alpha^4$ and $K_3 = \alpha^5$. By following the procedure of Theorem 9 we first create the

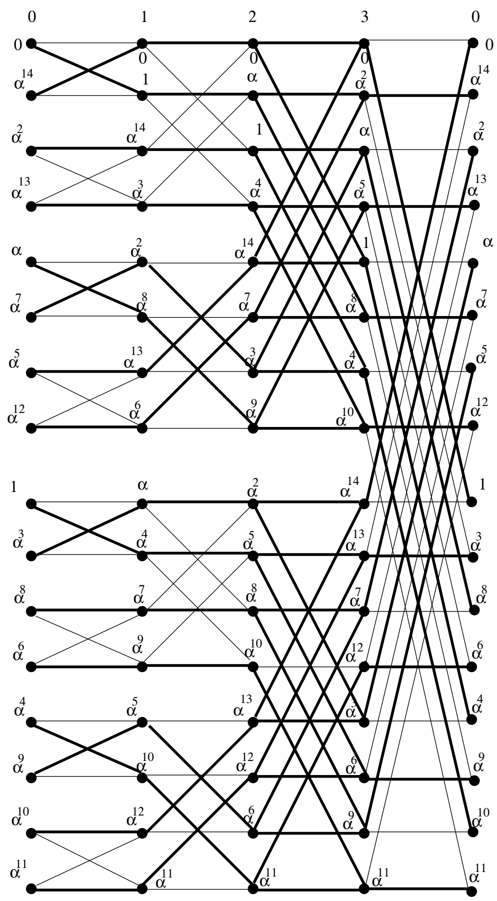

Fig. 1. Butterfly B_4 with faulty edges marked with light lines and fault-free edges with dark lines. The column numbers are at the top and the row index of each node is marked next to the node.

original Hamilton cycle as:

$$(0,1) \rightarrow (1,\alpha) \rightarrow (2,\alpha^2) \rightarrow (3,\alpha^3) \rightarrow (0,\alpha^4) \rightarrow$$
$$(1,\alpha^5) \rightarrow (2,\alpha^6) \rightarrow (3,\alpha^7) \rightarrow (0,\alpha^8) \rightarrow (1,\alpha^9) \rightarrow$$
$$(2,\alpha^{10}) \rightarrow (3,\alpha^{11}) \rightarrow (0,\alpha^{12}) \rightarrow (1,\alpha^{13}) \rightarrow (2,\alpha^{14}) \rightarrow$$
$$(3,0) \rightarrow (0,0) \rightarrow (1,0) \rightarrow (2,0) \rightarrow (3,1) \rightarrow$$
$$(0,\alpha) \rightarrow (1,\alpha^2) \rightarrow (2,\alpha^3) \rightarrow (3,\alpha^4) \rightarrow (0,\alpha^5) \rightarrow$$
$$(1,\alpha^6) \rightarrow (2,\alpha^7) \rightarrow (3,\alpha^8) \rightarrow (0,\alpha^9) \rightarrow (1,\alpha^{10}) \rightarrow$$
$$(2,\alpha^{11}) \rightarrow (3,\alpha^{12}) \rightarrow (0,\alpha^{13}) \rightarrow (1,\alpha^{14}) \rightarrow (2,1) \rightarrow$$
$$(3,\alpha) \rightarrow (0,\alpha^2) \rightarrow (1,\alpha^3) \rightarrow (2,\alpha^4) \rightarrow (3,\alpha^5) \rightarrow$$
$$(0,\alpha^6) \rightarrow (1,\alpha^7) \rightarrow (2,\alpha^8) \rightarrow (3,\alpha^9) \rightarrow (0,\alpha^{10}) \rightarrow$$
$$(1,\alpha^{11}) \rightarrow (2,\alpha^{12}) \rightarrow (3,\alpha^{13}) \rightarrow (0,\alpha^{14}) \rightarrow (1,1) \rightarrow$$
$$(2,\alpha) \rightarrow (3,\alpha^2) \rightarrow (0,\alpha^3) \rightarrow (1,\alpha^4) \rightarrow (2,\alpha^5) \rightarrow$$
$$(3,\alpha^6) \rightarrow (0,\alpha^7) \rightarrow (1,\alpha^8) \rightarrow (2,\alpha^9) \rightarrow (3,\alpha^{10}) \rightarrow$$
$$(0,\alpha^{11}) \rightarrow (1,\alpha^{12}) \rightarrow (2,\alpha^{13}) \rightarrow (3,\alpha^{14}) \rightarrow (0,1)$$

By applying the automorphism offsets already calculated, one can then obtain the required fault-free Hamilton cycle as:

$$(0,\alpha^6) \rightarrow (1,\alpha^9) \rightarrow (2,\alpha^{10}) \rightarrow (3,\alpha^{11}) \rightarrow (0,\alpha^{11}) \rightarrow$$
$$(1,\alpha^{11}) \rightarrow (2,\alpha^{12}) \rightarrow (3,\alpha^{13}) \rightarrow (0,\alpha^3) \rightarrow (1,\alpha) \rightarrow$$
$$(2,\alpha^2) \rightarrow (3,\alpha^3) \rightarrow (0,\alpha) \rightarrow (1,\alpha^8) \rightarrow (2,\alpha^9) \rightarrow$$
$$(3,\alpha^5) \rightarrow (0,\alpha^{13}) \rightarrow (1,\alpha^3) \rightarrow (2,\alpha^4) \rightarrow (3,\alpha^{10}) \rightarrow$$
$$(0,\alpha^{12}) \rightarrow (1,\alpha^6) \rightarrow (2,\alpha^7) \rightarrow (3,\alpha^8) \rightarrow (0,\alpha^7) \rightarrow$$
$$(1,\alpha^2) \rightarrow (2,\alpha^3) \rightarrow (3,\alpha^4) \rightarrow (0,\alpha^{10}) \rightarrow (1,\alpha^{12}) \rightarrow$$
$$(2,\alpha^{13}) \rightarrow (3,\alpha^{14}) \rightarrow (0,0) \rightarrow (1,1) \rightarrow (2,\alpha) \rightarrow$$
$$(3,\alpha^2) \rightarrow (0,\alpha^{14}) \rightarrow (1,0) \rightarrow (2,0) \rightarrow (3,0) \rightarrow$$
$$(0,1) \rightarrow (1,\alpha^4) \rightarrow (2,\alpha^5) \rightarrow (3,\alpha^6) \rightarrow (0,\alpha^9) \rightarrow$$
$$(1,\alpha^5) \rightarrow (2,\alpha^6) \rightarrow (3,\alpha^7) \rightarrow (0,\alpha^2) \rightarrow (1,\alpha^{14}) \rightarrow$$
$$(2,1) \rightarrow (3,\alpha) \rightarrow (0,\alpha^8) \rightarrow (1,\alpha^7) \rightarrow (2,\alpha^8) \rightarrow$$
$$(3,\alpha^9) \rightarrow (0,\alpha^5) \rightarrow (1,\alpha^{13}) \rightarrow (2,\alpha^{14}) \rightarrow (3,1) \rightarrow$$
$$(0,\alpha^4) \rightarrow (1,\alpha^{10}) \rightarrow (2,\alpha^{11}) \rightarrow (3,\alpha^{12}) \rightarrow (0,\alpha^6)$$

Theorem 9 is interesting because it implies that up to 2^{n-1} edges of the same type may be faulty in up to $n-1$ columns and the faulty butterfly is still Hamiltonian. It is easy to extend this idea to any other mapping also. A direct result of Theorem 9 is the following result.

Corollary 1: A butterfly with $n-1$ edge faults distributed one per column is Hamiltonian.

We now give an alternate simple proof due to the results developed in this paper of a previously a known result [14].

Theorem 10: Graph B_n with up to 2 random edge faults is Hamiltonian.

Proof. If there is only one fault or if there are two faults, both in the same type (f or g) of edges, or if they are in two different columns, then by Theorem 9 one can generate a Hamiltonian cycle for B_n. Thus we only need to treat cases that involve two faulty edges of different types (one f and one g) in the same column.

Consider now the case of an f and a g faulty edge in the same column m such that they do not share a node. Let the faulty f edge be $(m-1, X) \rightarrow (m, \alpha X)$, $X \neq 0$. In this case, one can first create a cycle containing all the nodes (i, X), $0 \leq i \leq n-1$, $X \neq 0$ using only the f edges. Clearly this cycle avoids the faulty g edge. Further, it can be easily modified to avoid the faulty f edge. To achieve this, add the g edges $(m-1, X) \rightarrow (m, \alpha X + \beta_{n-1})$ and $(m-1, X+\beta_0) \rightarrow (m, \alpha X)$ and remove the f edges $(m-1, X) \rightarrow (m, \alpha X)$ and $(m-1, X+\beta_0) \rightarrow (m, \alpha X + \beta_{n-1})$ as shown in Fig. 2. This removes the faulty f edge from the cycle, but partitions it into two disjoint cycles.

We now show that there exist g edges (shown as horizontal lines in Fig. 2) connecting the two parts which can be used to rejoin the two halves and create a single cycle of all the nodes (m, x), $x \neq 0$ without any faulty edge. It is easy to see that the number of nodes in each part is a multiple of n, and in fact, is at least $2n$. Let k be any integer between 0 and $n-1$ other than $m-1$ or $m-2 \mod n$. This is always possible because

$n \geq 3$. Since there are exactly $2^n - 1$ nodes with first index k in the two cycles, one of the cycles will have an odd number of such nodes. Without loss of generality, assume that it is the right cycle. Consider a typical node (k, y) in this cycle. If node $(k+1, \alpha y + \beta_{n-1})$ also belongs to the same cycle, then the g edge from $(k, y) \rightarrow (k+1, \alpha y + \beta_{n-1})$ will end up in the same cycle. At the same time, the node $(k, y + \beta_0)$ which belongs to the same cycle will have a g edge going to $(k+1, \alpha y)$ in the same cycle. Thus the g edges starting from that cycle and ending up in the same cycle occur in pairs. Since there are odd number of nodes with first index k, one of these nodes, say (k, Y), will have a g edge to the node $(k+1, \alpha y + \beta_{n-1})$ in the left cycle. Further, the node $(k, Y + \beta_0)$ from the left cycle has a g edge ending up at $(k+1, \alpha Y)$ in the right cycle. Using this pair of g edges, one can create a cycle of all nodes (i, x), $x \neq 0$ without using any faulty edge as shown in Fig. 2. Note that because the f and g edge faults are not incident on the same node, none of the g edges used here are faulty.

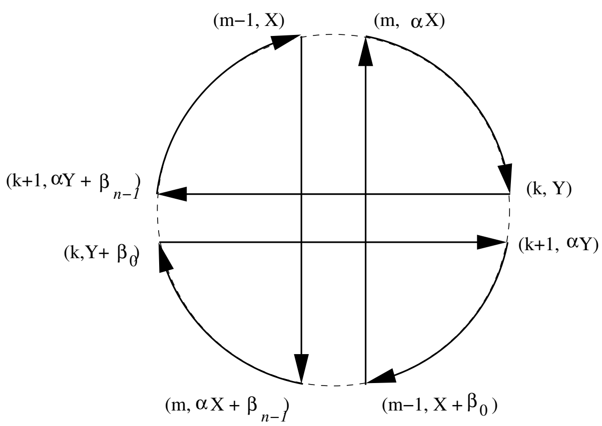

Fig. 2. Fault free cycle of all nodes (i, X), $0 \leq i \leq n-1$, $X \neq 0$ when an f edge $(m-1, X) \rightarrow (m, \alpha X)$ is faulty.

To add the rest of the B_n nodes to the Hamiltonian cycle, one can build the cycle of all the nodes $(i, 0)$, $0 \leq i \leq n-1$, using faultless f edges and merge it with the cycle in Fig. 2 using g edges in any column other than m.

If the faulty f edge is $(m-1, 0) \rightarrow (m, 0)$, then one can create a cycle of all nodes (t, X), $X \neq 0$ using faultless f edges, and of all nodes $(t, 0)$ using f edges. The faulty f edge will be in the second cycle. Merging the two cycles gets rid of the faulty edge.

Finally, consider the case of the faulty f and g edges in the same column and also sharing a node. We initially construct a Hamiltonian cycle considering that both f and g edges from the node $(0, 0)$ to be faulty. This cycle can then be translated using an appropriate automorphism to one that avoids f and g edges from any node.

We first partition the butterfly nodes into three sets of nodes connected by f edges as follows.

Set 1: $(1, 0) \xrightarrow{f} (2, 0) \xrightarrow{f} (3, 0) \xrightarrow{f} \cdots (0, 0)$.

Set 2: $(0, \beta_0) \xrightarrow{f} (1, \beta_{n-1}) \xrightarrow{f} (2, \alpha\beta_{n-1}) \xrightarrow{f} \cdots (n-1, \beta_0)$.

Set 3: $(0, \beta_{n-1}) \xrightarrow{f} (1, \alpha\beta_{n-1}) \xrightarrow{f} (2, \alpha^2\beta_{n-1}) \xrightarrow{f} \cdots (n-1, \alpha^{-1}\beta_0)$.

Note that sets 2 and 3 when joined together give the cycle of length $n2^n - n$ containing all the nodes with nonzero second coordinates obtained by continuously traveling along the f edges. Set 1 contains all the B_n nodes with their second coordinate 0. We can connect sets 1 and 2 into a cycle because their endpoints are connected by g edges. In particular, $(0, \beta_0) \xrightarrow{g} (1, 0)$ and $(n-1, \beta_0) \xrightarrow{g} (0, 0)$. The nodes in Set 3 can be incorporated in this cycle if the two end nodes of Set 3 are connected to some two consecutive nodes in the cycle. Note that the end nodes of Set 3 have the following connectivity: $(0, \beta_{n-1}) \xrightarrow{g} (1, \alpha\beta_{n-1} + \beta_{n-1})$ and $(n-1, \alpha^{-1}\beta_0) \xrightarrow{g} (0, \beta_0 + \beta_{n-1})$. One can verify that $(0, \beta_0 + \beta_{n-1}) \xrightarrow{f} (1, \alpha\beta_{n-1} + \beta_{n-1})$. Thus if node $(0, \beta_0 + \beta_{n-1})$ is in Set 2, then one can remove the f edge between this node and the next, and instead connect the nodes of Set 3 into the cycle using the g edges noted here. The resultant cycle is shown in Fig. 3.

On the other hand, if the node $(0, \beta_0 + \beta_{n-1})$ is in Set 3 rather than in Set 2, then the g edges from the endpoints of Set 3 go to adjacent nodes of Set 3, namely the nodes $(1, \alpha\beta_{n-1} + \beta_{n-1})$ and $(0, \beta_0 + \beta_{n-1})$. By removing the f edge between these adjacent nodes and adding the g edges from the endpoints, one can see that all the nodes of Set 3 form a cycle. To show that this cycle can be merged with the cycle formed by the nodes in Sets 1 and 2, we show that there is some (m, β_0) in cycle 3 with $m \neq 0$ and $m+1 \neq 0$. Because then, one can drop edge $(m, \beta_0) \xrightarrow{f} (m+1, \beta_{n-1})$ in the cycle of Set 3 and instead use connections to merge this cycle with Set 1 using edges $(m, \beta_0) \xrightarrow{g} (m+1, 0)$ and $(m, 0) \xrightarrow{g} (m+1, \beta_{n-1})$. To see that such a node (m, β_0) exist in Set 3, note that the number of nodes in Set 3 is at least $2^n - 1$. In other words, the second coordinate of the nodes in Set 3 take all possible nonzero values. Consequently, there will be some (m, β_0) present in Set 3. Further, both $(0, \beta_0)$ and $(n-1, \beta_0)$ are in Set 2, showing $m \neq 0, n-1$. Thus nodes in Set 3 can also be merged in the cycle formed by nodes in Sets 1 and 2. This gives the required Hamiltonian cycle. ∎

VI. Conclusion

In the past, automorphisms have been used to map algorithms on architectures with (generally one) node fault. This paper has shown that automorphisms can also be used to map algorithms on architectures with edge faults. To achieve this, we propose the use of an appropriate interconnection graph automorphism to map the set of faulty edges to free edges. However, in order to be able to use this strategy in different situations, one needs to know all the automorphisms and how each maps edges of the graph. This paper has obtained all the $n2^{n+1}$ automorphisms of the butterfly of dimension n. We use an algebraic model of the butterfly presented in [11].

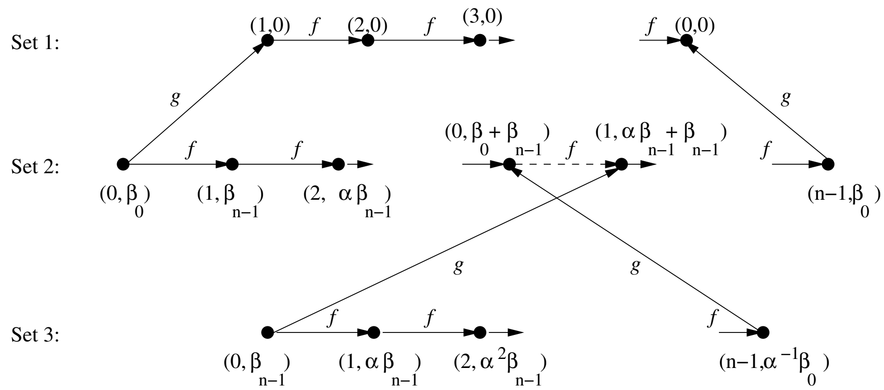

Fig. 3. The Hamiltonian cycle when the f and g edges from $(0, 0)$ are faulty and the node $(0, \beta_0 + \beta_{n-1})$ is in Set 2. Note that all edges are bidirectional and the dashed f edge is not part of the cycle.

The resultant automorphisms are simple; they map the two coordinates of a node label independently. This simplicity has allowed us to determine the mapping of edges due to any automorphism. We have illustrated our technique by mapping a Hamilton cycle on a butterfly under various edge fault scenarios. We believe that having a large set of $n2^{n+1}$ simple automorphisms, each with a specific determined edge translation property makes this method applicable to a large number of mappings on faulty butterflies.

REFERENCES

[1] F. Leighton, *Introduction to parallel algorithms and architectures: Arrays, trees, hypercubes*. M. Kaufman Pub., 1992.

[2] R. Feldmann and W. Unger, "The cube-connected-cycle is a subgraph of the butterfly network," *Parallel Processing Letters*, vol. 2, no. 1, pp. 13–19, 1992.

[3] O. Guzide and M. D. Wagh, "Extended butterfly networks," in *Proc. of the The 18th Int. Conf. on Parallel and Distributed Computing Systems*, Las Vegas, NV, 2005, pp. 109–113.

[4] ——, "Enhanced butterfly: A Cayley graph with node degree 5," in *Proc. of the The 20th Int. Conf. on Parallel and Distributed Computing Systems*, Las Vegas, NV, 2007, pp. 224–229.

[5] C. T. Gray, W. Liu, T. Hughes, and R. Cavin, "The design of a high-performance scalable architecture for image processing applications," in *Proc 90 Int. Conf. Appl. Specif. Array Processors*, 1991, pp. 722–733.

[6] W. Lin, T. Sheu, C. R. Das, T. Feng, and C. Wu, "Fast data selection and broadcast on the butterfly network," in *Proc Workshop Future Trends Distrib Comput Syst 1990s*, 1990, pp. 65–72.

[7] A. Ranade, "Optimal speedup for backtrack search on a butterfly network," *Mathematical systems theory*, vol. 27, no. 1, pp. 85–102, Jan 1994.

[8] J. H. Reif and S. Sen, "Randomized algorithms for binary search and load balancing on fixed connection networks with geometric applications," *SIAM Journal on Computing*, vol. 23, no. 3, pp. 633–651, Jun 1994.

[9] E. J. Schwabe, "Constant-slowdown simulations of normal hypercube algorithms on the butterfly network," *Information processing letters*, vol. 45, no. 6, p. 295, Apr 1993.

[10] S. Bhatt, F. R. K. Chung, J. Hong, F. Leighton, B. Obrenic, A. L. Rosenberg, and E. J. Schwabe, "Optimal emulation by butterfly-like networks," *JACM*, vol. 43, no. 2, pp. 293–330, Mar 1996.

[11] M. D. Wagh and O. Guzide, "Mapping cycles and trees on wrap-around butterfly graphs," *SIAM J. on Comput.*, vol. 35, pp. 741–765, 2006.

[12] P. Vadapalli and P. Srimani, "Fault tolerant ring embedding in tetravalent cayley network graphs," *J Circuits Syst Comput*, vol. 10, pp. 527–536, 1996.

[13] C.-H. Tsai, T. Liang, L.-H. Hsu, and M.-Y. Lin, "Cycle embedding in faulty wrapped butterfly graphs," *Networks*, vol. 42, no. 2, pp. 85–96, 2003.

[14] S.-C. Hwang and G.-H. Chen, "Cycles in butterfly graphs," *Networks*, vol. 35, no. 2, pp. 161–171, 2000.

[15] A. Avior, T. Calamoneri, S. Even, A. Litman, and A. L. Rosenberg, "A tight layout of the butterfly network," *Theory of Computing Systems*, vol. 31, no. 4, pp. 475–488, Dec 1998.

[16] C.-H. Yeh, B. Parhami, E. A. Varvarigos, and H. Lee, "VLSI layout and packaging of butterfly networks," in *Proc. of ACM symp. on Parallel algorithms and architectures*. New York, NY, USA: ACM, 2000, pp. 196–205.