MATH 243 PRACTICE EXAM 9 ON IDEALS AND THE FIRST ISOMORPHISM THEOREM FOR RINGS

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Let R be a nonzero ring and let $\phi : \mathbb{Z}/5\mathbb{Z} \to R$ be a ring homomorphism. Which of the following is necessarily true?

A. ϕ is not injective.

B. ϕ is injective.

C. ϕ is not surjective.

D. ϕ is surjective.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Which of the following elements does not belong to the ideal $(2, x)$ of $\mathbb{Z}[x]$?

A. 2.

B. x.

C. $2x$.

D. $2 - x$.

E. All of these belong to the ideal $(2, x)$.

Problem 3. (5 pts.) Circle the correct answer.

How many ideals does the ring $\mathbb{Z}/4\mathbb{Z}$ have?

A. 1.
B. 2.
C. 3.
D. 4.
E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

Let R be a nonzero commutative ring and let $I \neq (1)$ be a proper ideal of R. Which of the following is false?

A. I is closed under addition: $I + I \subseteq I$.
B. I is closed under R-addition: $R + I \subseteq I$.
C. I is closed under multiplication: $II \subseteq I$.
D. I is closed under R-multiplication: $RI \subseteq I$.
E. None of these.

Problem 5. (15 pts.) Find the number of elements in each of the following rings.

(a) $\mathbb{Z}/3\mathbb{Z}[x]/(x^3+2x^2)$.

(b) $\mathbb{Z}[x]/(2,x)$.

(c) $\mathbb{Z}[x]/(x^2+2)$.

(d) $\mathbb{Z}/5\mathbb{Z}[x]/(x^2+2)$.

(e) $\mathbb{Z}/7\mathbb{Z}[x]/(x^8)$.

Problem 6. (15 pts.) Find expressions for the following rings as quotients of familiar rings.

(a) $\mathbb{Z}/3\mathbb{Z}$ with a square root of 2 adjoined.

(b) $\mathbb{Z}/5\mathbb{Z}$ with a (new) square root of 0 adjoined.

(c) $\mathbb{R}$ with a square root of -1 adjoined (i.e., $\mathbb{C}$).

(d) $\mathbb{Z}$ with a multiplicative inverse of 2 adjoined.

(e) $\mathbb{Z}/7\mathbb{Z}$ with two new square roots of 3 adjoined.

Problem 7. (15 pts.) Expand and simplify $(1 + x)^6$ in $\mathbb{Z}/5\mathbb{Z}[x]$.

Problem 8. (15 pts.) In the ring $\mathbb{Z}/3\mathbb{Z}[x]/(x^3+2x^2+x+2)$, the element $(1+2x^2)^2$ is equal to what polyomial of degree at most 2?

Problem 9. (15 pts.) Let $\phi : R \to S$ be a ring homomorphism.

(a) Prove that $\ker(\phi)$ is not a subring of R unless S is the zero ring.

(b) Prove that $\phi(R)$ is a subring of S.

(c) Prove that $R\ker(\phi) = \ker(\phi)$.

Problem 10. (10 pts) Complete the following claim, whose proof is given.

Claim: Let $\phi : R \to S$ be a surjective ring homomorphism with kernel I. We have

Proof. Define the map

$$\begin{aligned} \overline{\phi} : R/I &\to S \\ a + I &\mapsto \phi(a). \end{aligned}$$

By the first isomorphism theorem for groups we know that $\overline{\phi} : (R/I, +) \to (S, +)$ is a group isomorphism. In particular, it is bijective. We claim that it is also a ring homomorphism. To see this, observe that for $a, b \in R$ we have

$$\begin{aligned} \overline{\phi}((a + I)(b + I)) &= \overline{\phi}(ab + I) = \phi(ab), \\ \overline{\phi}(a + I)\overline{\phi}(b + I) &= \phi(a)\phi(b) = \phi(ab), \end{aligned}$$

since ϕ is a ring homomorphism. We also have

$$\overline{\phi}(1 + I) = \phi(1_R) = 1_S,$$

again, since ϕ is a ring homomorphism. It follows that $\overline{\phi}$ is a bijective ring homomorphism, i.e., a ring isomorphism. □

Problem 11. (10 pts) Complete the following claim, whose proof is given.

Claim: Let $\phi : R$ be an integral domain. We have

Proof. ($\Rightarrow$) Suppose R is a field. Clearly (0) is an ideal of R. Choose $a \in R \setminus \{0\}$. Then a is a unit and the smallest ideal of R containing a also contains $aa^{-1} = 1$. Thus it is (1).

($\Leftarrow$) Suppose that the only ideals of R are (0) and (1). Choose $a \in R \setminus \{0\}$. We must have $(a) = (1)$ and therefore we can write $1 = ra$ for some $r \in R$. But then r must be the inverse of a. It follows that a is a field. $\square$