MATH 243 PRACTICE EXAM 7 ON NORMAL SUBGROUPS AND THE FIRST ISOMORPHISM THEOREM

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Let group G and subgroup $H \subseteq G$ satisfy $[G : H] = 2$. What can we conclude about the relationship of H and G?

A. H is not normal in G.

B. H is normal in G.

C. H is normal in G only if G is abelian.

D. H is normal in G only if G is nonabelian.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Let G be a subgroup of H, and let H be a subgroup of K. Which of the following statements about the three groups is true?

A. If G is normal in H, then G is normal in K.

B. If G is normal in K, then G is normal in H.

C. If H is normal in K, then G is normal in K.

D. If G is normal in K, then H is normal in K.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

Which of the following groups is isomorphic to $\mathbb{Z}/24\mathbb{Z}$?

A. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

B. $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

C. $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

D. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

A 5-hour clock and a 12-hour clock begin together. For which pairs (a, b) of integers with $0 \leq a < 5$ and $0 \leq b < 12$ will the clocks simultaneously read a o'clock and b o'clock, respectively?

A. None of these pairs.

B. Only the pairs with a dividing 12 and b dividing 5.

C. Only the pairs with a not dividing 12 and b not dividing 5.

D. All of these pairs.

E. Other.

Problem 5. (20 pts.) Prove that the alternating group A_n has exactly $n!/2$ elements, and therefore is normal in S_n.

Problem 6. (20 pts.) Given a group G, let $\mathrm{Inaut}(G)$ denote the group of inner automorphisms of G, and let $\mathrm{Aut}(G)$ denote the group of all automorphisms of G. Decide if it is true that for any group G, the group $\mathrm{Inaut}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Problem 7. (20 pts.) Let G be a group.

(a) Show that for every group H and every homomorphism $\phi : G \to H$, the subgroup $\ker \phi$ is a normal subgroup of G.

(b) Show that for every normal subgroup K of G, there exists a group H and a homomorphism $\phi : G \to H$ such that $K = \ker \phi$.

Problem 8. (20 pts.) Decide if in $\mathrm{GL}_n(\mathbb{R})$, the subgroup of diagonal matrices is normal.

Problem 9. (15 pts.) Let G be a group, and let

$$Z = Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$$

be the center of G. Show that Z is a subgroup of G and that it is in fact normal in G.

Problem 10. (15 pts.) Find a subgroup of the symmetric group S_4 which is not normal in S_4. Justify your answer.

Problem 11. (15 pts.) Show that the quotient $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ is isomorphic to $(\mathbb{R}, \cdot)$.