MATH 243 PRACTICE EXAM 6 ON HOMOMORPHISMS

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Let b be an element of a group G and define the map

$$\begin{aligned} \lambda_b : G &\to G, \\ g &\mapsto bg. \end{aligned}$$

Which of the following statements best describes λ_b?

A. It is neither bijective nor a homomorphism.

B. It is bijective but is not a homomorphism.

C. It is not bijective but is a homomorphism.

D. It is a bijective homomorphism.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Let b be an element of a group G and define the map

$$\begin{aligned} \gamma_b : G &\to G, \\ g &\mapsto bgb^{-1}. \end{aligned}$$

Which of the following statements best describes γ_b?

A. It is neither bijective nor a homomorphism.

B. It is bijective but is not a homomorphism.

C. It is not bijective but is a homomorphism.

D. It is a bijective homomorphism.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

Let G be a group and define the map

$$\begin{aligned} \mathrm{triv} : G &\to G, \\ g &\mapsto e. \end{aligned}$$

Which of the following statements best describes triv?

A. It is neither bijective nor a homomorphism.

B. It is bijective but is not a homomorphism.

C. It is not bijective but is a homomorphism.

D. It is a bijective homomorphism.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

Let G be a group and define the map

$$\begin{aligned} \mathrm{id} : G &\to G, \\ g &\mapsto g. \end{aligned}$$

Which of the following statements best describes id?

A. It is neither bijective nor a homomorphism.

B. It is bijective but is not a homomorphism.

C. It is not bijective but is a homomorphism.

D. It is a bijective homomorphism.

E. None of these.

Problem 5. (5 pts.) Circle the correct answer.

Let $G = (G, \cdot)$ be an abelian group and define the map

$$\begin{aligned} \text{invert} : G &\to G, \\ g &\mapsto g^{-1}. \end{aligned}$$

Assuming that not all elements of G are equal to their own inverses, which of the following statements best describes invert?

A. It is a homomorphism but is not an isomorphism.

B. It is an isomorphism but is not an automorphism.

C. It is an automorphism but is not an inner automorphism.

D. It is an inner automorphism.

E. None of these.

Problem 6. (5 pts.) Circle the correct answer.

Let G be a group and define the set

$$\text{Aut} = \{\phi : G \to G \,|\, \phi \text{ is an automorphism }\}.$$

Which of the following statements best describes Aut, with map composition?

A. It is not closed under map composition and thus is not a semigroup.

B. It is a semigroup but has no identity map and thus is not a monoid.

C. It is a monoid but does not have inverses and thus is not a group.

D. It is a group.

E. None of these.

Problem 7. (20 pts.) Let $\phi : G \to H$ be a homomorphism.

(a) Show that $\ker \phi$ is a subgroup of G.

(b) Show that $\mathrm{im}\phi$ is a subgroup of H.

(c) If $|G| = 12$ and $|H| = 8$, then what possible values can $|\ker \phi|$ have?

Problem 8. (20 pts.) Show that a homomorphism $\phi : G \to H$ is injective if and only if $\ker \phi = \{e_G\}$.

Problem 9. (20 pts.) Let G, H, K be groups. Show that if $\phi : G \to H$ is a homomorphism and if $\psi : H \to K$ is a homomorphism, then the composition

$$\begin{aligned} \psi\phi : G &\to K \\ g &\mapsto \psi(\phi(g)) \end{aligned}$$

is also a homomorphism.

Problem 10. (20 pts.) Show that $(\mathbb{Z}, +)$ is isomorphic to its proper subgroup $(2\mathbb{Z}, +)$.