MATH 243 PRACTICE EXAM 4 ON MATRIX GROUPS

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Which of the following is a subgroup of $\mathrm{SL}_n(\mathbb{Q})$?

A. $\mathrm{SL}_n(\mathbb{R})$.

B. $\mathrm{SO}_n(\mathbb{R})$.

C. $\mathrm{SO}_n(\mathbb{Q})$.

D. $\mathrm{GL}_n(\mathbb{Q})$.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

If $AA^T = I$ then which of the following best describes the determinant of A?

A. It is necessarily 1.

B. It is necessarily -1.

C. It could be 1 or -1.

D. It could be 1, 0, or -1.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

Which of the following is the intersection of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{Q})$?

A. $\mathrm{SO}_n(\mathbb{R})$.

B. $\mathrm{O}_n(\mathbb{R})$.

C. $\mathrm{SO}_n(\mathbb{Q})$.

D. $\mathrm{O}_n(\mathbb{Q})$.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

Define the permutation matrix

$$P = (p_{i,j}) = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

which is equal to the $n \times n$ identity matrix, except for entries $p_{1,1}$, $p_{1,2}$, $p_{2,1}$, $p_{2,2}$. Which of the following implications is true?

A. If $A \in \mathrm{GL}_n(\mathbb{R})$ then $PA \in \mathrm{GL}_n(\mathbb{R})$.

B. If $A \in \mathrm{SL}_n(\mathbb{R})$ then $PA \in \mathrm{SL}_n(\mathbb{R})$.

C. If $A \in \mathrm{O}_n(\mathbb{R})$ then $PA \in \mathrm{O}_n(\mathbb{R})$.

D. If $A \in \mathrm{SO}_n(\mathbb{R})$ then $PA \in \mathrm{SO}_n(\mathbb{R})$.

E. None of these.

Problem 5. (15 pts.) Show that $\mathrm{SO}_n(\mathbb{R})$ is a group.

Problem 6. (10 pts) Draw a diagram which shows the containment relations satisfied by the groups

$$\mathrm{GL}_n(\mathbb{R}), \quad \mathrm{SL}_n(\mathbb{R}), \quad \mathrm{O}_n(\mathbb{R}), \quad \mathrm{SO}_n(\mathbb{R}), \quad S_n, \quad A_n,$$

where S_n is the symmetric group, and A_n is the alternating group, both viewed as groups of matrices.

Problem 7. (15 pts.) Find a subgroup of $\mathrm{O}_n(\mathbb{R})$ which is isomorphic to the dihedral group D_{2n}.

Problem 8. (15 pts.) Consider the set $H = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid \det(A)^4 = 1\}$, and recall that ± 1 and $\pm i$ all are fourth roots of unity in $\mathbb{C}$. Show that H is a group.

Problem 9. (15 pts.) Show that the set $\mathrm{Mat}_{n\times n}(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{R})$ is not a group. What is the largest group inside of this set?