MATH 243 PRACTICE EXAM 3 ON THE SYMMETRIC GROUP

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

The permutation 52431 in S_5 has how many inversions?

A. 5.

B. 6.

C. 7.

D. 8.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

What is the cardinality of S_4?

A. 4.

B. 8.

C. 16.

D. 24.

E. None of these.

Problem 3. (15 pts.) Show that exactly half of the elements of the symmetric group belong to the alternating group.

Problem 4. (15 pts.) Find reduced expressions for the following elements of S_n.

(a) $s_1s_2s_1s_2s_1$.

(b) $s_1s_3s_1s_2s_1$.

(c) $s_1s_2s_3s_2s_1$.

(d) $s_1s_2s_3s_1s_2s_3s_1s_2s_3$.

Problem 5. (15 pts.) Let $\{s_1, \dots, s_6\}$ be the standard generators of the symmetric group S_7.

(a) Find a reduced (short as possible) expression for the permutation whose word one-line notation is 5274613.

(b) Find the word one-line notation of the permutation $s_1 s_2 s_3 s_4 s_1$.

(c) Find the cycle type of the permutation whose functional one-line notation is 2431675.

(d) Find the matrix notation of the permutation $s_1 s_3 s_4 s_6$.

Problem 6. (15 pts.) Recall that the symmetric group S_4 is generated by elements s_1, s_2, s_3 subject to relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \ldots, 3, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2, \end{aligned}$$

and that we define the *word one-line notation* $w_1w_2w_3w_4$ of any element $w = s_{i_1} \cdots s_{i_\ell}$ in S_4 to be $w_1w_2w_3w_4 = s_{i_1}(s_{i_1}(\cdots s_{i_\ell}(1234)\cdots))$, where s_i swaps the letters in positions i, $i+1$ of any four-letter word.

Solve the following problems for the element

$$w = s_3 s_2 s_1$$

in S_4 and show your work.

(a) Find the word one-line notation $w_1w_2w_3w_4$ of the element w.

(b) Find $\textsc{inv}(w)$, the number of inversions in $w_1w_2w_3w_4$.

(c) Use 2-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

to find the word one-line notation of w^2.

(continued $\rightarrow$)

(d) Find $\textsc{inv}(w^2)$.

(e) Find a reduced (short as possible) expression for w^2 in terms of s_1, s_2, s_3.

Problem 7. (10 pts) Complete the following claim, whose proof is given.

Claim: For $w \in S_n$ we have

Proof. ($\textsc{inv}(w) \leq \ell(w)$) Let $s_{i_1} \cdots s_{i_k}$ be a reduced expression for w and consider the word one-line notation

$$w_1 \cdots w_n = s_{i_1}(s_{i_2}(\cdots(1 \cdots n)\cdots)).$$

of w. Since the action of each generator on the word to its right increases the number of inversions by at most 1, the total number of inversions in $w_1 \cdots w_n$ can be no more than k.

($\ell(w) \leq \textsc{inv}(w)$) This is true in the symmetric group S_2 on 2 letters: e is an expression of length 0 and its word one-line notation 12 has 0 inversions; s_1 is an expression of length 1 and its word one-line notation 21 has 1 inversion.

Now suppose that the inequality holds for symmetric groups $S_2, \ldots, S_{n-1}$. Then by the first part of the proof, equality $\ell(w) = \textsc{inv}(w)$ in fact holds for w in these groups. Now consider $w = w_1 \cdots w_n \in S_n$. Suppose that the letter n appears in position $k+1$ and write $v = v_1 \cdots v_{n-1}$ for the other letters,

$$w = v_1 \cdots v_k n v_{k+1} \cdots v_{n-1}.$$

Let $p = \textsc{inv}(w) = \ell(v)$ and write an expression $s_{i_1} \cdots s_{i_p}$ for v. Then we have

$$\textsc{inv}(w) = p + (n-1) + k,$$

and one expression

$$w = s_{k+1} \cdots s_{n-1}(v_1 \cdots v_{n-1} n) = s_{k+1} \cdots s_{n-1} s_{i_1} \cdots s_{i_p}$$

for w. While we don't know this to be the *shortest* expression for w, we can conlude that the shortest expression for w is no longer than this. In other words, w has length at most $p + (n-1) + k = \textsc{inv}(w)$. ∎

Problem 8. (10 pts) Prove that the symmetric group S_n, viewed as a set of permutation matrices, is a subgroup of the real orthogonal group $\mathrm{O}_n(\mathbb{R})$. (We already know that S_n is a group; just prove that it is a subset of $\mathrm{O}_n(\mathbb{R})$.)

Problem 9. (10 pts) Show that the dihedral group D_{2n} can be viewed as a subgroup of the symmetric group S_n. (We already know that D_{2n} is a group; just explain how it can be viewed as a subset of S_n.)