MATH 243 PRACTICE EXAM 2 ON CYCLIC AND DIHEDRAL GROUPS

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Fix $n \geq 3$. How many symmetries of the regular n-gon are there?

A. n.

B. $2n$.

C. $n!$.

D. $2n!$.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Fix $n \geq 3$ and let D_{2n} be the dihedral group generated by ρ, τ subject to the relations

$$\rho^n = \tau^2 = e, \qquad \tau\rho = \rho^{n-1}\tau.$$

Which of the following statements best describes the subgroup generated by ρ?

A. It is a dihedral group of order n.

B. It is a cyclic group of order n.

C. It is a cyclic group of order $2n$.

D. It is a symmetric group of order $n!$.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

Let D_8 be the dihedral group of symmetries of the square, generated by ρ, τ subject to the relations

$$\rho^4 = \tau^2 = e, \qquad \tau\rho = \rho^3\tau.$$

Which of the following elements is equal to $\tau\rho^2\tau\rho^3\tau$?

A. τ.

B. $\rho\tau$.

C. $\rho^2\tau$.

D. $\rho^3\tau$.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

Let C_5 be the cyclic group of rotations of the regular pentagon, and let ρ be a generator of this group. Which of the following expressions is equal to ρ?

A. ρ^{-4}.

B. ρ^{-1}.

C. ρ^2.

D. ρ^3.

E. None of these.

Problem 5. (15 pts.)

Fix $n \geq 3$ and let $C_{16} = \{g, g^2, \ldots, g^{16} = e\}$ be the cyclic group of order 16. For any element $h \in C_{16}$, let $\langle h \rangle$ be the smallest subgroup of C_{16} containing h.

(a) For how many elements $h \in C_{16}$ is $\langle h \rangle$ equal to the entire group?

(b) For how many elements $h \in C_{16}$ is $\langle h \rangle$ a 4-element subgroup of C_{16}?

Problem 6. (15 pts.)

Fix $n \geq 3$ and let D_{2n} be the dihedral group generated by ρ, τ subject to the relations

$$\rho^n = \tau^2 = e, \qquad \tau\rho = \rho^{n-1}\tau.$$

(a) Show that for $k = 0, \ldots, n-1$, the element $\rho^k\tau$ has order 2.

(b) Show that D_{2n} is not abelian.

Problem 7. (15 pts.) Complete the proof that for $k, n \in \mathbb{P}$ and $d = \gcd(k, n)$, we have

$$k\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}.$$

Proof. Recall that $d = \alpha k + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$.

$(\supseteq)$

$(\subseteq)$

$\square$

Problem 8. (10 pts) Complete the following claim, whose proof is given.

Claim: Suppose that g generates the cyclic group C_n. Then

Proof. ($\Rightarrow$) Suppose $\gcd(k,n) \neq 1$. Then for some $d > 1$ we have $\gcd(k,n) = d$ and we may write $k = k'd$, $n = n'd$. Now consider the different powers of g^k. Since each is $(g^k)^r = (g^{dk'})^r = (g^d)^{k'r}$, we see that all such elements belong to the list

$$g^d, g^{2d}, g^{3d}, \ldots, g^{n'd} = g^n = e,$$

since g generates C_n. Since the exponents here are all multiples of d, $g = g^1$ is missing. If follows that g^k does not generate all of C_n.

($\Leftarrow$) Suppose $\gcd(k,n) = 1$. Then we can write $1 = \alpha k + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$, and

$$g = g^1 = g^{\alpha k + \beta n} = (g^k)^\alpha (g^n)^\beta = (g^k)^\alpha.$$

But then every element $g^r \in C_n$ can be written as

$$g^r = ((g^k)^\alpha)^r = (g^k)^{\alpha r},$$

Thus g^k generates all of C_n. $\square$