MATH 243 PRACTICE EXAM 1 ON UNIVERSAL ALGEBRA

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Which of the following statements best describes the algebraic properties of the set

$$\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$$

of nonzero real numbers, with ordinary multiplication?

A. $(\mathbb{R}^*, \times)$ is not a semigroup because it is not closed under $\times$.

B. $(\mathbb{R}^*, \times)$ is a semigroup but not a monoid, because it is closed under $\times$, but has no multiplicative identity element.

C. $(\mathbb{R}^*, \times)$ is a monoid but not a group, because it is closed under $\times$ and has a multiplicative identity element, but has no multiplicative inverse elements.

D. $(\mathbb{R}^*, \times)$ is a group, because it is closed under $\times$, has a multiplicative identity element, and has multiplicative inverse elements.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Which of the following statements best describes the algebraic properties of the set

$$\mathbb{Z}^* = \{x \in \mathbb{Z} \mid x \neq 0\}$$

of nonzero integers, with ordinary multiplication?

A. $(\mathbb{Z}^*, \times)$ is not a semigroup because it is not closed under $\times$.

B. $(\mathbb{Z}^*, \times)$ is a semigroup but not a monoid, because it is closed under $\times$, but has no multiplicative identity element.

C. $(\mathbb{Z}^*, \times)$ is a monoid but not a group, because it is closed under $\times$ and has a multiplicative identity element, but has no multiplicative inverse elements.

D. $(\mathbb{Z}^*, \times)$ is a group, because it is closed under $\times$, has a multiplicative identity element, and has multiplicative inverse elements.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

Which of the following statements best describes the algebraic properties of the set

$$(2\mathbb{Z}+1, \times)$$

of odd integers with ordinary multiplication?

A. $(2\mathbb{Z}+1, \times)$ is not a semigroup because it is not closed under $\times$.

B. $(2\mathbb{Z}+1, \times)$ is a semigroup but not a monoid, because it is closed under $\times$, but has no multiplicative identity element.

C. $(2\mathbb{Z}+1, \times)$ is a monoid but not a group, because it is closed under $\times$ and has a multiplicative identity element, but has no multiplicative inverse elements.

D. $(2\mathbb{Z}+1, \times)$ is a group, because it is closed under $\times$, has a multiplicative identity element, and has multiplicative inverse elements.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

Which of the following statements best describes the algebraic properties of the set

$$(2\mathbb{Z}, \times)$$

of even integers with ordinary multiplication?

A. $(2\mathbb{Z}, +)$ is not a semigroup because it is not closed under $+$.

B. $(2\mathbb{Z}, +)$ is a semigroup but not a monoid, because it is closed under $+$, but has no additive identity element.

C. $(2\mathbb{Z}, +)$ is a monoid but not a group, because it is closed under $+$ and has an additive identity element, but has no additive inverse elements.

D. $(2\mathbb{Z}, +)$ is a group, because it is closed under $+$, has a additive identity element, and has additive inverse elements.

E. None of these.

Problem 5. (15 pts.) Let $\mathrm{GL}_2(\mathbb{R}) = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid \det(A) \neq 0\}$ be the set of invertible 2×2 matrices, and let $\cdot$ denote matrix multiplication. Thus $(\mathrm{GL}_2(\mathbb{R}), \cdot)$ is a group.

(a) What is the multiplicative identity element of $(\mathrm{GL}_2(\mathbb{R}), \cdot)$?

(b) Explain why for each $A \in \mathrm{GL}_2(\mathbb{R})$ we also have that A^{-1} belongs to $\mathrm{GL}_2(\mathbb{R})$.

(c) Explain why for each pair $A, B \in \mathrm{GL}_2(\mathbb{R})$ we also have that AB belongs to $\mathrm{GL}_2(\mathbb{R})$.

(d) Show by example that $(\mathrm{GL}_2(\mathbb{R}), \cdot)$ is not an abelian group.

(continued $\rightarrow$)

(e) Let T be the subset of diagonal matrices in $\mathrm{GL}_2(\mathbb{R})$. This subset forms a group as well. Show that $(T, \cdot)$ is abelian.

(f) Show that $(\mathrm{GL}_2(\mathbb{R}), +)$ is not closed.

(g) Show that even $(T, +)$ is not closed.

Problem 6. (15 pts.) Show that if an algebra $(\mathcal{A}, \star)$ has a left-identity element e_L and a right-identity element e_R, i.e.,

$$e_L \star g = g, \qquad g \star e_R = g$$

for all $g \in A$, then we must have $e_L = e_R$.

Problem 7. (10 pts) Complete the following claim, whose proof is given.

Claim: Let $(\mathcal{A}, \star)$ be a monoid with identity element e. For every invertible element $g \in \mathcal{A}$ we have

Proof. Suppose that g has a left inverse and a right inverse,

$$h_L \star g = g \star h_R = e.$$

Since monoids are associative we may right "multiply" these expressions by h_R and may add parentheses to obtain

$$h_L \star (g \star h_R) = h_R,$$

and we may conclude that the expression on the left is equal to h_L and to h_R, i.e., these elements are equal to one another.

Now suppose that elements f, h are inverses of g. By the above argument, each of these elements is both a left-inverse and a right inverse of g. Also by the above argument, every left inverse of g is equal to every right inverse of g. Thus f and g are equal. $\square$

Problem 8. (15 pts.) Show that for all real numbers a, b, not both 0, the complex number $a + bi$ has multiplicative inverse given by

$$\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i.$$

Problem 9. (15 pts) Let $R(x)$ be the set of rational functions of x with coefficients in R, e.g.,

$$\mathbb{Q}(x) = \{\tfrac{p(x)}{q(x)} \mid p(x) \in \mathbb{Q}[x],\ q(x) \in \mathbb{Q}[x],\ q(x) \neq 0\}.$$

Show that $\mathbb{Q}(x) = \mathbb{Z}(x)$.