MATH 243 PRACTICE EXAM 10 ON VECTOR SPACES OVER AN ARBITRARY FIELD

Name:

Instructions:

(1) This is a closed-book, closed-notes, no-calculator exam.
(2) Show your work as instructed.

Do not write below this line.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

Total

Problem 1. (5 pts.) Circle the correct answer.

Let p be a prime number. How many elements belong to the vector space $(\mathbb{Z}/p\mathbb{Z})^k$?

A. p.

B. k.

C. p^k.

D. k^p.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer.

Let S be a set of 4 vectors in the rational function vector space $(\mathbb{R}(y))^3$. Which of the following statements about S is necessarily true?

A. S spans $(\mathbb{R}(y))^3$.

B. S does not span $(\mathbb{R}(y))^3$.

C. S is linearly independent in $(\mathbb{R}(y))^3$.

D. S is not linearly independent in $(\mathbb{R}(y))^3$.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer.

The linear system
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
in $\mathbb{Z}/3\mathbb{Z}$ has how many solutions?

A. 0.

B. 1.

C. 3.

D. infinitely many.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer.

What conditions on a prime number p guarantee that the matrix
$$\begin{bmatrix} 1 & 4 & 0 \\ 4 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$$
belongs to $\mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$?

A. $p \geq 3$ only.

B. $p \geq 5$ only.

C. $p \geq 7$ only.

D. $p \geq 11$ only.

E. None of these.

Problem 5. (5 pts.) Circle the correct answer.

What is the dimension of the ring $\mathbb{Z}/13\mathbb{Z}[x]/(x^4 + x^3 + x^2)$ as a vector space over $\mathbb{Z}/13\mathbb{Z}$?

A. 13^2.

B. 13^3.

C. 13^4.

D. 13^5.

E. None of these.

Problem 6. (5 pts.) Circle the correct answer.

Define the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}$$

in $\mathrm{GL}_3(\mathbb{Z}/7\mathbb{Z})$. What is $\det(A)$?

A. 1.

B. 2.

C. 3.

D. 6.

E. None of these.

Problem 7. (15 pts.) Use Gaussian elimination to solve the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

in $\mathbb{Z}/5\mathbb{Z}$.

Problem 8. (15 pts.) Invert the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

in $\mathrm{Mat}_{3\times 3}(\mathbb{Z}/7\mathbb{Z})$, or show that it is not invertible.

Problem 9. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let F be a field, let A belong to $\mathrm{Mat}_{m\times n}(F)$, let $\mathbf{b}$ belong to F^m, and let $\mathbf{c}, \mathbf{d}$ belong to F^n. If $\mathbf{x} = \mathbf{c}$ is a solution of the system

and $\mathbf{x} = \mathbf{d}$ is a solution of the system

then $\mathbf{x} = \mathbf{c} + \mathbf{d}$ is a solution of the system

Proof. Suppose that we have $A\mathbf{c} = \mathbf{b}$ and $A\mathbf{d} = \mathbf{0}$. By the distributive law we then have

$$A(\mathbf{c} + \mathbf{d}) = A\mathbf{c} + A\mathbf{d} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

$\square$

Problem 10. (15 pts.) Define the $\mathbb{Z}/2\mathbb{Z}$-vector space and the $\mathbb{Z}/3\mathbb{Z}$-vector space

$$V_2 = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{Z}/2\mathbb{Z}) \mid A^\top = -A\},$$
$$V_3 = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{Z}/3\mathbb{Z}) \mid A^\top = -A\}.$$

Show that $\dim(V_2) = 6$ and $\dim(V_3) = 3$. Find a basis for each space.

Problem 11. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let F be a field and let V be an n-dimensional F-vector space. Then

Proof. Let $\{v_1, \ldots, v_n\}$ be a linearly independent set of n vectors in V. Choose $w \in V$. We claim that w belongs to $\operatorname{span}_F\{v_1, \ldots, v_n\}$. To see this, observe that the $(n+1)$-element set $\{v_1, \ldots, v_n, w\}$ is linearly dependent: it has too many elements to be linearly independent. Thus we can write

$$c_1 v_1 + \cdots + c_n v_n + c_{n+1} w = 0_V$$

with scalars $c_1, \ldots, c_{n+1}$ not all 0. In particular, c_{n+1} cannot be 0, because this would produce a non-trivial linear combination $c_1 v_1 + \cdots + c_n v_n$ equal to 0_V, contradicting the linear independence of $\{v_1, \ldots, v_n\}$. Therefore we can express w as

$$w = -c_{n+1}^{-1} c_1 v_1 - \cdots - c_{n+1}^{-1} c_n v_n,$$

which implies that w belongs to $\operatorname{span}_F\{v_1, \ldots, v_n\}$. Thus the set $\{v_1, \ldots, v_n\}$ spans V. Since it is also linearly independent, it is a basis of V. $\square$

Problem 12. (15 pts.) Let F be a field, let $B = (v_1, \dots, v_n)$ be a basis of an F-vector space V, and let w be any vector in V. Show that if there are scalars $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying

$$w = a_1v_1 + \cdots + a_nv_n, \qquad w = b_1v_1 + \cdots + b_nv_n,$$

then we must have $a_i = b_i$ for $i = 1, \dots, n$.