MATH 242 PRACTICE EXAM ON EIGENVALUES, EIGENVECTORS, AND DIAGONALIZATION

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the matrix A given by

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Which of the following statements about the eigenspaces of A is true?

A. The 3-eigenspace has dimension 0.

B. The 4-eigenspace has dimension 1.

C. The 5-eigenspace has dimension 2.

D. The 12-eigenspace has dimension 3.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a matrix and let $\mathbf{x}$ be an eigenvector of A with eigenvalue λ. Which of the following statements about A^2 is true?

A. $\mathbf{x}$ is an eigenvector of A^2 with eigenvalue λ.

B. $\mathbf{x}^2$ is an eigenvector of A^2 with eigenvalue λ.

C. $\mathbf{x}$ is an eigenvector of A^2 with eigenvalue λ^2.

D. $\mathbf{x}^2$ is an eigenvector of A^2 with eigenvalue λ^2.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a matrix with the property that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector of A with eigenvalue 6. What can we say about the entries of A?

A. The entries of each column sum to 6.

B. The entries of each row sum to 6.

C. The determinant of A is 6.

D. The trace of A is 6.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 4×4 matrix having distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. What is the determinant of A?

A. $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

B. $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$.

C. $\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$.

D. $\lambda_1\lambda_2\lambda_3\lambda_4$.

E. None of these.

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $T : V \to V$ be a linear map. How are the 0-eigenspace and kernel of T related?

A. The kernel and 0-eigenspace intersect, but neither contains the other.

B. The kernel is properly contained in the 0-eigenspace.

C. The 0-eigenspace is properly contained in the kernel.

D. The kernel and 0-eigenspace are equal.

E. None of these.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let P_3 be the space of polynomials of z of degree at most 3. Define the linear map

$$T : P_3 \to P_3$$
$$p(z) \mapsto \frac{d}{dz}((1+z)p(z)).$$

The eigenvector $(1+z)^2$ of T has what eigenvalue?

A. 0.

B. 1.

C. 2.

D. 3.

E. None of these.

Problem 7. (10 pts.) Find the eigenvalues of the matrix

$$\begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$$

and one nonzero eigenvector for each.

Problem 8. (20 pts.) Define the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which has characteristic polynomial $-(\lambda - 2)^2(\lambda + 1)$.

(a) Find the eigenvalues of A.

(b) Find the eigenspaces corresponding to each eigenvalue of A.

(continued →)

(c) Find a matrix Q such that $Q^{-1}AQ$ equals a diagonal matrix Λ (equivalently, $A = Q\Lambda Q^{-1}$), or explain why none exists.

(d) Find the diagonal matrix Λ (if it exists).

Problem 9. (20 pts.) Let α, β be real numbers with $\beta \neq 0$ and define the matrix

$$A = \begin{bmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{bmatrix},$$

which has characteristic polynomial $-(\lambda - \alpha - \beta)^2(\lambda - \alpha + 2\beta)$.

(a) Find the eigenvalues of A.

(b) Find the eigenspaces corresponding to each eigenvalue of A.

(continued $\rightarrow$)

(c) What are the dimensions of the eigenspaces that you found?

Problem 10. (25 pts.) Let P_2 be the space of polynomials in z of degree at most 2. Define the linear transformation

$$\begin{aligned} T : P_2 &\to P_2 \\ f(z) &\mapsto f(2z+1), \end{aligned}$$

i.e.,

$$T(a + bz + cz^2) = a + b(2z+1) + c(2z+1)^2.$$

(a) Find the matrix A of T with respect to the natural basis $\mathcal{B} = (1, z, z^2)$ of P_2.

(b) Find the eigenvalues of A and T.

(continued →)

(c) Find the eigenvectors of A.

(continued $\rightarrow$)

(d) Find a basis $\mathcal{B}'$ of V with respect to which the matrix of T is a diagonal matrix Λ, or explain why this is not possible.

(e) Find a matrix Q such that the matrix Λ above equals $Q^{-1}AQ$, or explain why this is not possible.

Problem 11. (5 pts.) Let $\mathfrak{sl}_2$ be the space of 2×2 matrices having trace 0,

$$\mathfrak{sl}_2 = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \,|\, m_{1,1} + m_{2,2} = 0\},$$

and define the linear transformation

$$\begin{aligned} T : \mathfrak{sl}_2 &\to \mathfrak{sl}_2 \\ M &\mapsto \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right] M - M \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]. \end{aligned}$$

(a) Find the matrix A of T with respect to the natural basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

of $\mathfrak{sl}_2$.

(b) Find the eigenvalues of A and T.

(continued $\to$)

(c) Find the eigenvectors of A.

(continued $\rightarrow$)

(d) Find a basis $\mathcal{B}'$ of V with respect to which the matrix of T is a diagonal matrix Λ, or explain why this is not possible.

(e) Find a matrix Q such that the matrix Λ above equals $Q^{-1}AQ$, or explain why this is not possible.

Problem 12. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let $T : V \to V$ be a linear transformation with nonzero eigenvectors $v_1, \ldots, v_k$ having distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, respectively. Then

Proof. Let v_1, v_2 be nonzero eigenvectors of T having eigenvalues $\lambda_1 \neq \lambda_2$. Suppose $\{v_1, v_2\}$ is linearly dependent. Then we have $v_2 = cv_1$ for some scalar $c \neq 0$. It follows that we have

$$T(v_2) = \lambda_2 v_2$$

and

$$T(v_2) = T(cv_1) = \lambda_1 c v_1 = \lambda_1 v_2 \neq \lambda_2 v_2,$$

a contradiction.

Now assume that the claim is true for collections of $2, \ldots, k-1$ eigenvectors with distinct eigenvalues, and consider a collection $\{v_1, \ldots, v_k\}$ of eigenvectors having distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Suppose $\{v_1, \ldots, v_k\}$ is linearly dependent. By assumption, $\{v_1, \ldots, v_{k-1}\}$ is linearly independent. Thus we have $v_k = c_1 v_1 + \cdots + c_{k-1} v_{k-1}$ for some scalars $c_1, \ldots, c_{k-1}$, not all 0. It follows that we have

$$T(v_k) = \lambda_k v_k = c_1 \lambda_k v_1 + \cdots + c_{k-1} \lambda_k v_{k-1}, \tag{1}$$

and also

$$T(v_k) = T(c_1 v_1 + \cdots + c_{k-1} v_{k-1}) = c_1 \lambda_1 v_1 + \cdots + c_{k-1} \lambda_{k-1} v_{k-1}. \tag{2}$$

Subtracting (1) from (2), we obtain

$$0 = c_1(\lambda_1 - \lambda_k) v_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1}. \tag{3}$$

Since $\{v_1, \ldots, v_{k-1}\}$ is a linearly independent set, each coefficient $c_j(\lambda_j - \lambda_k)$ in (3) must be zero. Since $\lambda_j \neq \lambda_k$ for all j, we must have $c_j = 0$ for all j. But this contradicts the fact that $c_1, \ldots, c_{k-1}$ are not all 0. Thus $\{v_1, \ldots, v_k\}$ cannot be a linearly dependent set. $\square$

Problem 13. (15 pts.) Let $T : V \to V$ be a linear transformation, and let A, A' be the matrices of T with respect to bases $\mathcal{B}$, $\mathcal{B}'$ of V, respectively. Prove that A and A' have the same eigenvalues.