MATH 242 PRACTICE EXAM ON LINEAR TRANSFORMATIONS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differentiation map $\frac{d}{dx} : P_3 \to P_2$ from the space of polynomials of degree at most 3 to the space of polynomials of degree at most 2, and define bases

$$\mathcal{B} = (1, x, x^2, x^3), \qquad \mathcal{C} = (1, x, x^2)$$

of P_3 and P_2, respectively. What is the matrix of $\frac{d}{dx}$ with respect to $(\mathcal{B}, \mathcal{C})$?

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

B. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

C. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

D. $\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}, \qquad g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x-y \end{bmatrix}, \qquad h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y-2x \end{bmatrix}.$$

Which of these maps are linear transformations?

A. f and g only.

B. f and h only.

C. g only.

D. h only.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $T : V \to \mathbb{R}$ be a linear transformation and suppose that $\mathbf{u}$ and $\mathbf{v}$ in V satisfy $T(\mathbf{v}) = 6$ and $T(\mathbf{u}) = -1$. What is $T(2\mathbf{v} - 3\mathbf{u})$?

A. 5.

B. 15.

C. 37.

D. 39.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let V be a 2-dimensional vector space and let linear transformation $T : V \to V$ have matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

with respect to some basis $\mathcal{B}$. Which of the following properties does T have?

A. It is neither injective nor surjective.

B. It is injective but not surjective.

C. It is surjective but not injective.

D. It is bijective.

E. None of these.

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V, let $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$, and let $T : V \to V$ be a linear transformation. If A is the matrix of T with respect to $(\mathcal{B}, \mathcal{B})$, then what is the matrix of T with respect to $(\mathcal{C}, \mathcal{C})$?

A. PAP.

B. PAP^{-1}.

C. $P^{-1}AP$.

D. $P^{-1}AP^{-1}$.

E. None of these.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V, let $T : V \to V$ be a linear transformation, and let A, A' be the matrices of T with respect to $(\mathcal{B}, \mathcal{B})$, and $(\mathcal{C}, \mathcal{C})$, respectively. Which of the following statements best describes common properties of A and A'?

A. $\det(A) = \det(A')$.

B. $\text{nullity}(A) = \text{nullity}(A')$.

C. $\text{rank}(A) = \text{rank}(A')$.

D. All of the above.

E. None of these.

Problem 7. (15 pts.) Define the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 2y - z \\ x + 3y \\ 2z \end{bmatrix}.$$

(a) Find the matrix representation of T with respect to the standard basis of $\mathbb{R}^3$.

Problem 8. (20 pts.) Define the map

$$\begin{aligned} T : P_2 &\to P_2 \\ f(x) &\mapsto f(x) - xf'(x). \end{aligned}$$

(a) Show that T is linear.

(b) Find a basis for the image of T.

(continued →)

(c) Show that T is or is not injective.

(d) Show that T is or is not surective.

Problem 9. (20 pts.) Let M_2 be the space of all 2×2 matrices with real entries, and define the map

$$\begin{aligned} T : M_2 &\to M_2 \\ B &\mapsto B + B^{\top}. \end{aligned}$$

(a) Show that T is linear.

(b) Find the kernel of T and its dimension.

(continued $\rightarrow$)

(c) Find the range of T and its dimension.

(d) State an equation which relates $\dim(\ker(T))$ and $\dim(\mathrm{range}(T))$ to $\dim(M_2)$.

Problem 10. (15 pts.) Define the three-dimensional space

$$V = \operatorname{span}\{\sin^2(x),\, \cos^2(x),\, \sin(x)\cos(x)\}.$$

(a) Show that the derivatives of $\sin^2(x)$, $\cos^2(x)$, $\sin(x)\cos(x)$ all belong to V.

(b) By part (a), and since differention is linear, we have a linear map $\frac{d}{dx} : V \to V$. Find the matrix of $\frac{d}{dx}$ with respect to the basis $(\,\sin^2(x),\, \cos^2(x),\, \sin(x)\cos(x)\,)$.

Problem 11. (20 pts.) Define the linear map

$$\begin{aligned} T : P_2 &\to P_2 \\ p(x) &\mapsto (x+2)p'(x) \end{aligned}$$

(a) Find the matrix A of T with respect to the natural basis $\mathcal{B} = (1, x, x^2)$ of P_2.

(b) Find the matrix M of T with respect to the basis $\mathcal{C} = (1, x+2, (x+2)^2)$ of P_2.

(continued $\rightarrow$)

(c) Find an invertible matrix Q such that A and M are related by $M = Q^{-1}AQ$.

Problem 12. Complete the following claim, whose proof is given.

Claim: The image and kernel of a linear transformation $T : V \to W$ satisfy

Proof. Choose bases $\mathcal{B}$, $\mathcal{C}$ of V, W, and let A be the matrix of T with respect to $(\mathcal{B}, \mathcal{C})$. Then A is an $m \times n$ matrix, where $m = \dim W$ and $n = \dim V$.

Since $\operatorname{im} T$ is isomorphic to the column space of A, and $\ker T$ is isomorphic to the nullspace of A, we have

$$\dim(\operatorname{im} T) = \operatorname{rank}(A), \qquad \dim(\ker T) = \operatorname{nullity}(A).$$

From the rank-nullity theorem,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n,$$

we thus have the claimed equation. $\square$