MATH 242 PRACTICE EXAM ON NULLSPACES, COLUMN SPACES, AND ROW SPACES.

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 3×4 matrix. What can we say about the dimensions of its row space and column space?

A. $\dim(\text{rowspace}(A)) < \dim(\text{colspace}(A))$.

B. $\dim(\text{rowspace}(A)) = \dim(\text{colspace}(A))$.

C. $\dim(\text{rowspace}(A)) \neq \dim(\text{colspace}(A))$.

D. $\dim(\text{rowspace}(A)) > \dim(\text{colspace}(A))$.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an invertible 3×3 matrix. What is the nullity of A?

A. 0.

B. 1.

C. 2.

D. 3.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 4×4 matrix, let $\mathbf{b}$ be a vector in $\mathbb{R}^4$, and assume that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What is the rank of A?

A. 0.

B. 2.

C. 4.

D. 16.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 5×5 matrix with rank 4, and let $\mathbf{b}$ be a vector in $\mathbb{R}^5$. How many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have?

A. 0.

B. 1.

C. Infinitely many.

D. It depends if $\mathbf{b}$ belongs to the column space of A or not.

E. None of these.

Problem 5. (15 pts.) Let A be a 3×6 matrix,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \end{bmatrix},$$

and assume that performing row operations on A yields the matrix

$$B = \begin{bmatrix} 1 & 3 & 0 & 2 & 6 & -1 \\ 0 & 0 & 1 & 5 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find the dimension of the nullspace of A.

Problem 6. (20 pts.) Define the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 1 & -6 & 7 \end{bmatrix}.$$

(a) Find a basis of the row space of A.

(continued →)

(b) Find a basis of the column space of A.

(c) Find a basis of the nullspace of A.

Problem 7. (15 pts.) Let A be a 3×4 matrix, let $\mathbf{x}$ be a vector of 4 variables, and let $\mathbf{b}$ be a vector in $\mathbb{R}^3$. Suppose that one solution of the linear system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix},$$

and that a basis of the nullspace of A is

$$\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \right).$$

Describe all solutions of the linear system $A\mathbf{x} = \mathbf{b}$.

Problem 8. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let A be an $m \times n$ matrix, and let E be any matrix in row eschelon form which can be obtained by performing row operations to A. Then we have

Proof. Applying row operations to A is equivalent to left-multiplying by an invertible $m \times m$ matrix P. Thus we have $E = PA$.

($\subseteq$) Suppose that $\mathbf{v}$ belongs to the rowspace of A. Then for some scalars $c_1, \ldots, c_m$ we have

$$\mathbf{v} = c_1(\text{row } 1 \text{ of } A) + \cdots + c_m(\text{row } m \text{ of } A).$$

Equivalently,

$$\mathbf{v} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} \begin{bmatrix} \text{row } 1 \text{ of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} A.$$

But then we have

$$\begin{aligned} \mathbf{v} &= \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} P^{-1}PA \\ &= (\begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} P^{-1})E \\ &= \begin{bmatrix} d_1 & \cdots & d_m \end{bmatrix} E \end{aligned}$$

for scalars $d_1, \ldots, d_m$ satisfying $[d_1 \cdots d_m]P^{-1} = [c_1 \cdots c_m]$. It follows that $\mathbf{v}$ belongs to the rowspace of E.

($\supseteq$) Similarly, if $\mathbf{v}$ belongs to the rowspace of $E = PA$, then for some scalars $d_1, \ldots, d_m$ we have

$$\begin{aligned} \mathbf{v} &= \begin{bmatrix} d_1 & \cdots & d_m \end{bmatrix} PA \\ &= (\begin{bmatrix} d_1 & \cdots & d_m \end{bmatrix} P)A \\ &= \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} A, \end{aligned}$$

for scalars $c_1, \ldots, c_m$ satisfying $[c_1 \cdots c_m]P = [d_1 \cdots d_m]$. It follows that $\mathbf{v}$ belongs to the rowspace of A. $\square$