MATH 242 PRACTICE EXAM ON COORDINATE VECTORS AND CHANGE OF BASIS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let elements $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of a vector space V form a basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, and let $\mathbf{v} \in V$ satisfy $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for real numbers c_1, c_2, c_3. What is the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$?

A. $c_1c_2c_3$.

B. $c_1 + c_2 + c_3$.

C. $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

D. $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

E. Other.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let elements $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of a vector space V form a basis $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. What is the coordinate vector $[\mathbf{u}_1]_{\mathcal{B}}$ of $\mathbf{u}_1$ with respect to $\mathcal{B}$?

A. 0.

B. 1.

C. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

D. $\mathbf{u}_1$.

E. Other.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

What element of $\mathbb{R}^3$ has coordinate vector
$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$
with respect to the basis
$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)?$$

A. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

B. $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

C. $\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$.

D. $\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$.

E. Other.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

The polynomial $2 - 3x - 7x^2$ has coordinate vector

$$\begin{bmatrix} 2 \\ -3 \\ -7 \end{bmatrix}$$

with respect to what basis of P_2?

A. $(1, x, x^2)$.

B. $(2, -3x, -7x^2)$.

C. $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

D. $\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} \right)$.

E. Other.

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V. How are the change-of-basis matrices $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and $P_{\mathcal{C}\leftarrow\mathcal{B}}$ related?

A. $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

B. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{\top}$.

C. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$.

D. $P_{\mathcal{C}\leftarrow\mathcal{B}} = ((P_{\mathcal{B}\leftarrow\mathcal{C}})^{\top})^{-1}$.

E. None of these.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ be bases of a vector space V. Which of the following describes the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$?

A. Its jth column is the coordinate vector of $\mathbf{w}_j$ with respect to $\mathcal{B}$.

B. Its jth column is the coordinate vector of $\mathbf{v}_j$ with respect to $\mathcal{C}$.

C. Its jth column is the coordinate vector of $\mathbf{v}_j$ with respect to $\mathcal{B}$.

D. Its jth column is the coordinate vector of $\mathbf{w}_j$ with respect to $\mathcal{C}$.

E. None of these.

Problem 7. (20 pts.) Define the two bases

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right), \qquad \mathcal{C} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} \right)$$

of the space of symmetric 2×2 matrices $V = \{A \in M_2 \mid A^\top = A\}$.

(a) Find the change of basis matrix $P_{\mathcal{B} \leftarrow \mathcal{C}}$.

(b) Find the change of basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

(c) Find the coordinate vector of the matrix

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix}$$

with respect to $\mathcal{B}$.

(d) Find the coordinate vector of the matrix

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix}$$

with respect to $\mathcal{C}$.

Problem 8. (15 pts.) Define the three bases

$$\begin{aligned}\mathcal{B} &= (1, \quad x, \quad x^2),\\ \mathcal{C} &= (1, \quad 1+x, \quad 1+x+x^2),\\ \mathcal{C}' &= (1+x+x^2, \quad x+x^2, \quad x^2)\end{aligned}$$

of P_2.

(a) Find the change of basis matrix $P_{\mathcal{C}'\leftarrow\mathcal{C}}$. You may use the facts that

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}, \qquad \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix},$$

and $P_{\mathcal{C}'\leftarrow\mathcal{C}} = P_{\mathcal{C}'\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}$.

(b) If polynomial $p(x)$ has coordinate vector

$$\begin{bmatrix}5\\ 8\\ 1\end{bmatrix}$$

with respect to $\mathcal{C}$, then what is its coordinate vector with respect to $\mathcal{C}'$?

Problem 9. (20 pts.) Define the basis

$$\mathcal{C} = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right)$$

of $\mathbb{R}^3$. If $\mathbf{v}$ is the point

$$\begin{bmatrix} 21 \\ 7 \\ 15 \end{bmatrix}$$

in $\mathbb{R}^3$ (expressed in terms of the natural basis of unit vectors), then what is the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{C}$?

Problem 10. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let V be an n-dimensional vector space, with bases $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$, related by

(1)

where P is an $n \times n$ matrix. Then P must be

Proof. Suppose that P is not invertible. Then its columns are linearly dependent, and some column j equals a linear combination of the others,

$$\begin{bmatrix} \text{col} \\ j \\ \text{of} \\ P \end{bmatrix} = \sum_{i \neq j} d_i \begin{bmatrix} \text{col} \\ i \\ \text{of} \\ P \end{bmatrix}.$$

By (1) we then have

$$\mathbf{v}_j = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{bmatrix} \begin{bmatrix} \text{col} \\ j \\ \text{of} \\ P \end{bmatrix} = \sum_{i \neq j} d_i \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{bmatrix} \begin{bmatrix} \text{col} \\ i \\ \text{of} \\ P \end{bmatrix} = \sum_{i \neq j} d_i \mathbf{v}_i.$$

But this implies that $\mathcal{B}$ is a linearly dependent set, contradicting the fact that it is a basis. Therefore P must be invertible. $\square$