MATH 242 PRACTICE EXAM ON VECTOR SPACES AND BASES

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (18 pts.) For each of the following sets, write a check mark ✓ after it if it is a vector space; write an X after it if it is not a vector space. Let P_k denote the set of polynomials of degree at most k in a single variable x, having real coefficients.

There will be no partial credit. You do not need to show your work.

(i) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \text{ are even integers} \right\}.$

(ii) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$

(iii) $\{p(x) \in P(3) \,|\, \deg(p(x)) = 3\}.$

(iv) $\{p(x) \in P_3 \,|\, p(0) = 0\}.$

(v) $\{p(x) \in P_3 \,|\, p(1) = 0\}.$

(vi) $\{p(x) \in P_3 \,|\, p(0) = 1\}.$

Problem 2. (18 pts.) For each of the following sets, write a check mark ✓ after it if it is a vector space; write an X after it if it is not a vector space. Let M_n denote the set of $k \times k$ matrices having real entries.

There will be no partial credit. You do not need to show your work.

(i) $\{A \in M_2 \,|\, a_{1,1} + a_{2,2} = 0\}$.

(ii) $\{A \in M_2 \,|\, \det(A) \neq 0\}$.

(iii) $\left\{A \in M_k \,|\, A^{\top} = A\right\}$.

(iv) $\left\{A \in M_k \,|\, A^{\top} = -A\right\}$.

(v) $\left\{ \begin{bmatrix} a-b \\ a \\ a+b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$.

(vi) $\left\{ \begin{bmatrix} a-1 \\ a \\ a+1 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the three sets

$$S_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\},$$

$$S_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \right\},$$

$$S_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

Which of the sets is linearly independent?

A. S_1 only.

B. S_2 only.

C. S_1 and S_3 only.

D. S_2 and S_3 only.

E. Other.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Adjoining

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

to the sets from the previous problem, we obtain the three sets

$$T_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$T_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$T_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Which of these sets spans $\mathbb{R}^3$?

A. T_1 only.

B. T_2 only.

C. T_1 and T_3 only.

D. T_2 and T_3 only.

E. Other.

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \{1, \quad 1+x, \quad 1+x+x^2, \quad x+x^2, \quad x^2\}$$

of P_2. Which of the following best describes S?

A. It is a basis of P_2.

B. It spans P_2 but is not linearly independent.

C. It is linearly independent but does not span P_2.

D. It neither spans P_2 nor is linearly independent.

E. Other.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

of M_2. Which of the following best describes S?

A. It is a basis of M_2.

B. It spans M_2 but is not linearly independent.

C. It is linearly independent but does not span M_2.

D. It neither spans M_2 nor is linearly independent.

E. Other.

Problem 7. (15 pts.) For each of the sets below, show that it is linearly independent, or find a nontrivial linear combination of the elements which equals 0.

(a)

$$\left\{ \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \right\}.$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c)

$$\left\{1, \quad 1+x, \quad (1+x)^2\right\}.$$

(d)

$$\left\{1+x, \quad 1-x, \quad 1+x^2, \quad 1-x^2\right\}.$$

Problem 8. (25 pts.) For each of the following sets, show that it is or is not closed under addition and scalar multiplication. Then give one natural basis (or state that it is not a vector space), and find its dimension as a real vector space (or answer "DNA" if the set is not a vector space).

(a) The set $P_3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$.

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

(continued →)

(b) The subset V_1 of P_3 consisting of polynomials whose coefficients sum to 0,

$$V_1 = \{a + bx + cx^2 + dx^3 \in P_3 \,|\, a + b + c + d = 0\}.$$

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

(continued $\rightarrow$)

(c) The subset V_2 of V_1 consisting of polynomials having constant term equal to the coefficient of x^2 and coefficient of x equal to the coefficient of x^3,

$$V_2 = \{a + bx + cx^2 + dx^3 \in P_3 \,|\, a + b + c + d = 0,\ a = c,\ b = d\}.$$

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

Problem 9. (10 pts.) Consider the linearly independent set R and the spanning set T of $\mathbb{R}^3$ given by

$$R = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \qquad T = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} \right\}.$$

Find a basis S of $\mathbb{R}^3$ satisfying $R \subset S \subset T$. Briefly justify your answer.

Problem 10. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let V be an n-dimensional vector space. Then

Proof. Let $\{v_1, \dots, v_n\}$ be a linearly independent set of n vectors in V. Choose $w \in V$. We claim that w belongs to $\text{span}(v_1, \dots, v_n)$. To see this, observe that the $(n+1)$-element set $\{v_1, \dots, v_n, w\}$ is linearly dependent: it has too many elements to be linearly independent. Thus we can write

$$c_1 v_1 + \cdots + c_n v_n + c_{n+1} w = 0_V$$

with scalars $c_1, \dots, c_{n+1}$ not all 0. In particular, c_{n+1} cannot be 0, because this would produce a non-trivial linear combination $c_1 v_1 + \cdots + c_n v_n$ equal to 0_V, contradicting the linear independence of $\{v_1, \dots, v_n\}$. Therefore we can express w as

$$w = -\tfrac{c_1}{c_{n+1}} v_1 - \cdots - \tfrac{c_n}{c_{n+1}} v_n,$$

which implies that w belongs to $\text{span}(v_1, \dots, v_n)$. Thus the set $\{v_1, \dots, v_n\}$ spans V. Since it is also linearly independent, it is a basis of V. $\square$

Problem 11. (15 pts.) Let $B = (v_1, \dots, v_n)$ be a basis of a vector space V, and let w be any vector in V. Show that if there are scalars $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying

$$w = a_1 v_1 + \cdots + a_n v_n, \qquad w = b_1 v_1 + \cdots + b_n v_n,$$

then we must have $a_i = b_i$ for $i = 1, \dots, n$.