MATH 242 PRACTICE EXAM ON THE DETERMINANT

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the invertible matrices

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 4 \\ 7 & 5 \end{bmatrix}.$$

Which of the inverses A^{-1}, B^{-1}, C^{-1} have entries which are all integers?

A. A^{-1} only.

B. A^{-1} and B^{-1} only.

C. A^{-1}, B^{-1} and C^{-1}.

D. B^{-1} only.

E. Other.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 5×5 matrix having determinant 3. Which of the following is the determinant of $2A$?

A. 3.

B. 6.

C. 30.

D. 96.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Define the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Which of the following is equal to $\det(A)$?

A. 0.

B. 3.

C. 12.

D. 81.

E. Other.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. If $\det(A) = 2$, then how many solutions does the system $A\mathbf{x} = \mathbf{b}$ have?

A. 1.

B. 2.

C. $2n$.

D. 2^n.

E. None of these.

Problem 5. (15 pts.) Let A and B be 3×3 matrices satisfying

$$\det(A) = 3, \qquad \det(B) = -4.$$

Let A' be the matrix obtained from A by multiplying its second row by 3, and by replacing the third row by (first row of A + third row of A).

Let B' be the matrix obtained from B by multiplying its first column by 4 and by swapping its second and third columns.

Compute the determinants of the following matrices.

(a) $2A$.

(b) A^{-1}.

(c) $A^{\top}B$.

(d) B^5.

(e) $(B^{-1}AB)^2$.

(f) A'.

(g) B'.

Problem 6. (15 pts.) Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 1 & 1 & 0 & -2 \\ 6 & 0 & -1 & -2 \\ -3 & 1 & 3 & 2 \end{bmatrix}.$$

Show your work.

(a) $\det(A) =$

(a) $\det(B) =$

Problem 7. (15 pts.) Use the adjoint method to invert the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 5 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Show your work.

Problem 8. (10 pts.) Complete the following claim, whose proof is given.

Given an $n \times n$ matrix $A = (a_{i,j})$, let $M_{i,j} = M_{i,j}(A)$ be the determinant of the submatrix of A obtained by removing row i and column j,

$$M_{i,j} = \det(A_{\{1,\ldots,n\}\smallsetminus i, \{1,\ldots,n\}\smallsetminus j}).$$

Claim: For any $n \times n$ matrix A, we have

Proof. Consider the jth term in the sum on the right-hand side: $(-1)^{j-1} a_{1,j} M_{1,j}$. This is

$$(-1)^{j-1} a_{1,j} \sum_{\substack{w_2 \cdots w_n \\ \text{a permutation of} \\ \{1,\ldots,n\}\smallsetminus j}} (-1)^{\mathrm{INV}(w_2 \cdots w_n)} a_{2,w_2} \cdots a_{n,w_n}.$$

The sign of $a_{1,j} a_{2,w_2} \cdots a_{n,w_n}$ in this expression is

$$(-1)^{j-1} (-1)^{\mathrm{INV}(w_2 \cdots w_n)},$$

and as j varies from 1 to n, the word $j w_2 \cdots w_n$ varies over all permutations of $\{1, \ldots, n\}$.

Now consider the left-hand side: $\det(A)$ is a sum of products $\pm a_{1,w_1} \cdots a_{n,w_n}$ over all permutations $w_1 \cdots w_n$ of $\{1, \ldots, n\}$, and the sign of $a_{1,j} a_{2,w_2} \cdots a_{n,w_n}$ in $\det(A)$ is

$$(-1)^{\mathrm{INV}(j w_2 \cdots w_n)} = (-1)^{j-1} (-1)^{\mathrm{INV}(w_2 \cdots w_n)}.$$

Thus we have the desired equality. $\square$