MATH 242 PRACTICE EXAM ON JORDAN FORM

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Which of the following statements regarding an $n \times n$ matrix A and the $n \times n$ matrices $Q^{-1}AQ$, called *similar to* A, is true?

A. A is always similar to a diagonal matrix.

B. If A has repeated eigenvalues, then it is not similar to a diagonal matrix.

C. A is always similar to a Jordan matrix.

D. If A has distinct eigenvalues, then it is not similar to a Jordan matrix.

E. None of these.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the matrix A given by

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

What is the dimension of the (-2)-eigenspace of A?

A. 0.

B. 1.

C. 2.

D. 3.

E. None of these.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 6×6 matrix having characteristic polynomial $(\lambda+1)^3(\lambda-1)^2(\lambda-3)$, and let d be the dimension of the 1-eigenspace of A. What can we say about d?

A. $d = 2$.

B. $d = 3$.

C. $1 \leq d \leq 2$.

D. $1 \leq d \leq 3$.

E. None of these.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let c be a scalar and define the matrices

$$A = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}, \quad B = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}, \quad C = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{bmatrix}, \quad D = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{bmatrix}.$$

Which matrix has a 3-dimensional c-eigenspace?

A. A.

B. B.

C. C.

D. D.

E. None of these.

Problem 5. (20 pts.) Let P_2 be the space of polynomials in z of degree at most 2. Define the linear transformation

$$\begin{aligned} T : P_2 &\to P_2 \\ f(z) &\mapsto f(z+1), \end{aligned}$$

i.e.,

$$T(a + bz + cz^2) = a + b(z+1) + c(z+1)^2.$$

(a) Find the matrix A of T with respect to the natural basis $\mathcal{B} = (1, z, z^2)$ of P_2.

(b) Find the eigenvalues of A and T.

(c) Find the eigenvectors of A.

(continued →)

(d) Find a basis $\mathcal{B}'$ of V with respect to which the matrix of T is a Jordan matrix J, or explain why this is not possible.

(e) Find a matrix Q such that the matrix J above equals $Q^{-1}AQ$, or explain why this is not possible. Show J.

Problem 6. (20 pts.) Let $\mathfrak{sl}_2$ be the space of 2×2 matrices having trace 0,

$$\mathfrak{sl}_2 = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid m_{1,1} + m_{2,2} = 0\},$$

and define the linear transformation

$$\begin{aligned} T : \mathfrak{sl}_2 &\to \mathfrak{sl}_2 \\ M &\mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right](M - M^\top) - (M - M^\top)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \end{aligned}$$

i.e.,

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 2c - 2b & 0 \\ 0 & 2b - 2c \end{bmatrix}.$$

(a) Find the matrix A of T with respect to the natural basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

of $\mathfrak{sl}_2$.

(b) Show that 0 is the only eigenvalue of T and that the 0-eigenspace of A is two-dimensional with basis

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

(continued →)

(c) Let $\mathbf{x}$, $\mathbf{y}$ be the eigenvectors of A above. Show that only one of the systems $A\mathbf{w} = \mathbf{x}$, $A\mathbf{w} = \mathbf{y}$ has a solution.

(d) Use the basis $\mathcal{B}$ and the above vectors in $\mathbb{R}^3$ to find a basis $\mathcal{B}'$ of $\mathfrak{sl}_2$ with respect to which the matrix of T is a Jordan matrix J. Show J.

(e) Find a matrix Q such that the matrix J above equals $Q^{-1}AQ$, or explain why this is not possible.

Problem 7. (10 pts.) Complete the following claim, whose proof is given.

Claim: Let $T : V \to V$ be a linear transformation, let A be the matrix of T with respect to any basis of V, and let c be an eigenvalue of T which appears with multiplicity k in the characteristic equation $\det(A - \lambda I) = 0$, and let d be the dimension of the c-eigenspace of T. Then we have

Proof. We know that $d \geq 1$ because d is the dimension of the nullspace of the matrix $A - cI$, and the condition $\det(A - cI) = 0$ guarantees this dimension to be nonzero. Now choose a basis $(v_1, \ldots, v_d)$ of the c-eigenspace of T, and extend this to a basis $(v_1, \ldots, v_d, w_{d+1}, \ldots, w_n)$ of V. The matrix of T with respect to this basis is

$$A' = \begin{bmatrix} cI & M \\ 0 & N \end{bmatrix},$$

for I the $d \times d$ identity matrix, M a $d \times (n-d)$ matrix, and N an $(n-d) \times (n-d)$ matrix. Then we have $\det(A' - \lambda I) = (c - \lambda)^d \det(N - \lambda I)$. Since the multiplicity of c as a root of $\det(N - \lambda I) = 0$ is at least 0, it follows that k, the multiplicity of c as a root of $\det(A' - \lambda I) = 0$, is at least d. $\square$