MATH 205 PRACTICE EXAM ON EIGENVALUES AND EIGENVECTORS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the matrix A given by

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Which of the following statements about the eigenspaces of A is true?

A. The 3-eigenspace has dimension 0.

B. The 4-eigenspace has dimension 1.

C. The 5-eigenspace has dimension 2.

D. The 12-eigenspace has dimension 3.

E. None of these.

Answer: **B**.
Reason: Since the matrix A is upper triangular, the characteristic polynomial is given by $p(\lambda) = (3 - \lambda)(4 - \lambda)(5 - \lambda)$, meaning A has 3 distinct eigenvalues. This implies A has 3 distinct one-dimensional eigenspaces.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the matrix B given by

$$B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

What is the dimension of the (-2)-eigenspace of B?

A. 0.

B. 1.

C. 2.

D. 3.

E. None of these.

Answer: **B**.
Reason: Observe that

$$B + 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies the solution space of $(B + 2I)\mathbf{v} = \mathbf{0}$ is spanned by $\mathbf{v}_1 = (1, 0, 0)$, but there are no other eigenvectors.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a matrix with the property that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector of A with eigenvalue 6. What can we say about the entries of A?

A. The entries of each column sum to 6.

B. The entries of each row sum to 6.

C. The determinant of A is 6.

D. The trace of A is 6.

E. None of these.

Answer: **B**.
Reason: Observe that if $A = [\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4]$, then because the given vector is an eigenvector with eigenvalue 6, we have

$$\begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4.$$

This translates to the entries of each row summing to 6. (the sum of the first entries of each column, the second entries of each column, and so on.)

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 4×4 matrix having distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. What is the determinant of A?

A. $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

B. $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$.

C. $\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$.

D. $\lambda_1\lambda_2\lambda_3\lambda_4$.

E. None of these.

Answer: **D**.
Reason: The determinant of a matrix is always the product of its eigenvalues (you can prove this by using the definition of the characteristic polynomial). When A is diagonalizable (as it is when A has four distinct eigenvalues), then we can find an invertible matrix S so that $A = SDS^{-1}$ with $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Then

$$\det(A) = \det(SDS^{-1}) = \det(S)\det(D)\det(S)^{-1} = \det(D) = \lambda_1\lambda_2\lambda_3\lambda_4.$$

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a matrix and let $\mathbf{x}$ be an eigenvector of A with eigenvalue λ. Which of the following statements about A^2 is true?

A. $\mathbf{x}$ is an eigenvector of A^2 with eigenvalue λ.

B. $\mathbf{x}^2$ is an eigenvector of A^2 with eigenvalue λ.

C. $\mathbf{x}$ is an eigenvector of A^2 with eigenvalue λ^2.

D. $\mathbf{x}^2$ is an eigenvector of A^2 with eigenvalue λ^2.

E. None of these.

Answer: **C**.
Reason:

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 6×6 matrix having characteristic polynomial $(\lambda+1)^3(\lambda-1)^2(\lambda-3)$, and let d be the dimension of the 1-eigenspace of A. What can we say about d?

A. $d = 2$.

B. $d = 3$.

C. $1 \leq d \leq 2$.

D. $1 \leq d \leq 3$.

E. None of these.

Answer: **C**.
Reason: The geometric multiplicity of an eigenvalue (i.e. dimension the corresponding eigenspace) is always at least 1 and always less than or equal to the algebraic multiplicity of the eigenvalue (i.e. the number of times the root appears in the characteristic polynomial). Since the algebraic multiplicity of $\lambda = 1$ is 2, we must have $1 \leq d \leq 2$.

Problem 7. (20 pts.) Define the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which has characteristic polynomial $-(\lambda-2)^2(\lambda+1)$.

(a) Find the eigenvalues of A.

Solution: Since the characteristic polynomial has roots $\lambda_1 = 2$ and $\lambda_2 = -1$, these are the eigenvalues of A.

(b) Find the eigenspaces corresponding to each eigenvalue of A.

Solution: We first solve $(A-2I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A-2I|\mathbf{0}] = \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking x_2, x_3 as free variables, we see from the above that solution space of $(A-2I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_1 = (-1,1,0)$ and $\mathbf{v}_2 = (1,0,1)$. Therefore the 2-eigenspace is given by

$$E_{\lambda_1} = \operatorname{span}\Big\{(-1,1,0),(1,0,1)\Big\}.$$

Next we solve $(A+I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A+I|\mathbf{0}] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking x_3 as a free variable, we see from the above that solution space of $(A+I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_3 = (-1,-1,1)$. Therefore the (-1)-eigenspace is given by

$$E_{\lambda_2} = \operatorname{span}\Big\{(-1,-1,1)\Big\}.$$

(continued →)

(c) Find a matrix Q such that $Q^{-1}AQ$ equals a diagonal matrix Λ (equivalently, $A = Q\Lambda Q^{-1}$), or explain why none exists.

Solution: In the previous part, we found that A has 3 linearly independent eigenvectors $\mathbf{v}_1 = (-1, 1, 0), \mathbf{v}_2 = (-1, 0, 1), \mathbf{v}_3 = (-1, -1, 1)$. To diagonalize A, we take

$$Q = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(d) Find the diagonal matrix Λ (if it exists).

Solution: Because of the order of the eigenvectors in Q, the diagonal matrix Λ must be given by the corresponding eigenvalues:

$$\Lambda = \operatorname{diag}(2, 2, -1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is not necessary to do this on an exam, but, if you leftover time, you can confirm your answer by computing that

$$Q^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix},$$

and then checking that

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = Q^{-1}AQ.$$

Problem 8. (20 pts.) Define the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

Solution: Since the characteristic polynomial for the upper triangular matrix is given by $p(\lambda) = -(\lambda - 2)^2(\lambda - 4)$, the eigenvalues are given by $\lambda_1 = 2$ and $\lambda_2 = 4$.

(b) Find the eigenspaces corresponding to each eigenvalue of A.

Solution: We first solve $(A - 2I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking x_1 as free variable, we see from the above that solution space of $(A - 2I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_1 = (1, 0, 0)$. Therefore the 2-eigenspace is given by

$$E_{\lambda_1} = \text{span}\Big\{(1, 0, 0)\Big\}.$$

Even though the algebraic multiplicity of the eigenvalue 2 is two, the corresponding eigenspace only has dimension one.

Next we solve $(A - 4I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A - 4I|\mathbf{0}] = \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

Taking x_3 as a free variable, we see from the above that solution space of $(A - 4I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_2 = (3, -2, 4)$. Therefore the 4-eigenspace is given by

$$E_{\lambda_2} = \text{span}\Big\{(3, -2, 4)\Big\}.$$

You could naively take $\mathbf{v}_2 = (3/4, -1/2, 1)$ above by taking the free variable equal to 1 in your solution set, but I think it's nicer to work with integers, so I took the free variable x_3 equal to 4 to get rid of the fractions.

(continued →)

(c) Find a matrix Q such that $Q^{-1}AQ$ equals a diagonal matrix Λ (equivalently, $A = Q\Lambda Q^{-1}$), or explain why none exists.

Solution: No such Q exists because the matrix A is defective. That is, A does not have 3 linearly independent eigenvectors because the geometric multiplicity of the 2-eigenspace is only 1.

(d) Find the diagonal matrix Λ (if it exists).

Solution: It does not exist.

Problem 9. (20 pts.) Let α, β be real numbers with $\beta \neq 0$ and define the matrix

$$A = \begin{bmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{bmatrix},$$

which has characteristic polynomial $-(\lambda - \alpha - \beta)^2(\lambda - \alpha + 2\beta)$.

(a) Find the eigenvalues of A.

Solution: Since the characteristic polynomial has roots $\lambda_1 = \alpha+\beta$ and $\lambda_2 = \alpha-2\beta$, these are the eigenvalues of A.

(b) Find the eigenspaces corresponding to each eigenvalue of A.

Solution: We first solve $(A-(\alpha+\beta)I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A - (\alpha+\beta)I|\mathbf{0}] = \left[\begin{array}{ccc|c} -\beta & -\beta & -\beta & 0 \\ -\beta & -\beta & -\beta & 0 \\ -\beta & -\beta & -\beta & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \beta & \beta & \beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

where we have used that $\beta \neq 0$ to divide by β in the last step. Taking x_2, x_3 as free variables, we see from the above that solution space of $(A-(\alpha+\beta)I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$. Therefore the $(\alpha+\beta)$-eigenspace is given by

$$E_{\lambda_1} = \operatorname{span}\Big\{(-1,1,0), (-1,0,1)\Big\}.$$

Next we solve $(A - (\alpha - 2\beta)I)\mathbf{x} = \mathbf{0}$. Row reducing the augmented matrix gives

$$[A + I|\mathbf{0}] = \left[\begin{array}{ccc|c} 2\beta & -\beta & -\beta & 0 \\ -\beta & 2\beta & -\beta & 0 \\ -\beta & -\beta & 2\beta & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking x_3 as a free variable, we see from the above that solution space of $(A - (\alpha - 2\beta)I)\mathbf{x} = \mathbf{0}$ is spanned by $\mathbf{v}_3 = (1, 1, 1)$. Therefore the $(\alpha - 2\beta)$-eigenspace is given by

$$E_{\lambda_2} = \operatorname{span}\Big\{(1,1,1)\Big\}.$$

(continued $\rightarrow$)

(c) What are the dimensions of the eigenspaces that you found?

Solution: In part (b), we found that the $(\alpha + \beta)$-eigenspace has dimension 2 and the $(\alpha - 2\beta)$-eigenspace has dimension 1.

Problem 10. (20 pts.) Find the eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} 5 & 3 \\ -3 & 5 \end{bmatrix}.$$

Show your work.

Solution:

Step 1: Find eigenvalues. We have

$$p(\lambda) = \det(A - \lambda I) = (5-\lambda)^2 + 9 = \lambda^2 - 10\lambda + 34.$$

We find the roots by the quadratic formula:

$$\lambda = \frac{10 \pm \sqrt{100 - 4 \cdot 34}}{2} = 5 \pm \sqrt{-9} = 5 \pm 3i$$

The eigenvalues are therefore $\lambda_1 = 5 + 3i$ and $\lambda_2 = 5 - 3i$. Note that because A has real entries, but complex eigenvalues, the eigenvectors will come as complex conjugate pair.

Step 2: Find eigenvectors. We find the solutions to $(A - (5+3i)I)\mathbf{v} = \mathbf{0}$. We row reduce the augmented matrix

$$\begin{aligned}
[A - (5+3i)I|\mathbf{0}] = \left[\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right] &\sim \left[\begin{array}{cc|c} 3 & 3i & 0 \\ -3 & -3i & 0 \end{array}\right] && \text{(multiply row 1 by } i,\ M_1(i)) \\
&\sim \left[\begin{array}{cc|c} 3 & 3i & 0 \\ 0 & 0 & 0 \end{array}\right] && \text{(add rows, } A_{12}(1)) \\
&\sim \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] && \text{(multiply row 1 by } 1/3,\ M_1(1/3))
\end{aligned}$$

Take $x_2 = t$ to be the free variable. Then by the first row above we have $x_1 + i * x_2 = 0$ which implies $x_1 = i * t$. Therefore the solution space is spanned by (take $t = 1$), $\mathbf{v}_1 = (i, 1)$. Then, because λ_1 and λ_2 are complex conjugates, we must have that $\mathbf{v}_2 = (-i, 1)$ (the complex conjugate of $\mathbf{v}_1$) is an eigenvector of A with eigenvalue λ_2. We conclude

$$E_{\lambda_1} = \text{span}\{(i,1)\}, \qquad E_{\lambda_2} = \text{span}\{(-i,1)\}$$

are the eigenspaces of A.