MATH 205 PRACTICE EXAM ON LINEAR TRANSFORMATIONS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)	
(2)	
(3)	
(4)	
(5)	
(6)	
Total	

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differentiation map $\frac{d}{dx} : P_3 \to P_2$ from the space of polynomials of degree at most 3 to the space of polynomials of degree at most 2, and define bases

$$\mathcal{B} = \{1, x, x^2, x^3\}, \qquad \mathcal{C} = \{1, x, x^2\}$$

of P_3 and P_2, respectively. What is the matrix of $\frac{d}{dx}$ with respect to $(\mathcal{B}, \mathcal{C})$?

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

B. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

C. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

D. $\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

E. None of these.

Answer: **B**.
Reason: The matrix representation of $\frac{d}{dx}$ with respect to the bases $\mathcal{B}, \mathcal{C}$ is given by the matrix

$$\left[[\frac{d}{dx}1]_{\mathcal{C}}, [\frac{d}{dx}x]_{\mathcal{C}}, [\frac{d}{dx}x^2]_{\mathcal{C}}, [\frac{d}{dx}x^3]_{\mathcal{C}} \right] = \left[[0]_{\mathcal{C}}, [1]_{\mathcal{C}}, [2x]_{\mathcal{C}}, [3x^2]_{\mathcal{C}} \right].$$

This is the matrix in **B**.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}, \qquad g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x-y \end{bmatrix}, \qquad h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y-2x \end{bmatrix}.$$

Which of these maps are linear transformations?

A. f and g only.

B. f and h only.

C. g only.

D. h only.

E. None of these.

Answer: **D**.
Reason: Recall a linear map must satisfy $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$. In particular, one must have $T(\mathbf{0}) = \mathbf{0}$ and $T(-\mathbf{v}) = -T(\mathbf{v})$. Now we quickly see that f doesn't send the zero vector to the zero vector, so it cannot be linear. And g doesn't send negatives to negatives, so it cannot be linear. The last map is linear since it is can be expressed as a matrix multiplication $h(\mathbf{x}) = A\mathbf{x}$. Can you see what matrix A?

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $T : V \to \mathbb{R}$ be a linear transformation and suppose that $\mathbf{u}$ and $\mathbf{v}$ in V satisfy $T(\mathbf{v}) = 6$ and $T(\mathbf{u}) = -1$. What is $T(2\mathbf{v} - 3\mathbf{u})$?

A. 5.

B. 15.

C. 37.

D. 39.

E. None of these.

Answer: **B**.
Reason:

$$T(2\mathbf{v} - 3\mathbf{u}) = 2T(\mathbf{v}) - 3T(\mathbf{u}) = 2 \cdot 6 - 3 \cdot (-1) = 15.$$

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let V be a 2-dimensional vector space and let linear transformation $T : V \to V$ have matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

with respect to some basis $\mathcal{B}$. Which of the following properties does T have?

A. It is neither injective (into) nor surjective (onto).

B. It is injective (into) but not surjective (onto).

C. It is surjective (onto) but not injective (into).

D. It is bijective (into and onto).

E. None of these.

Answer: **D**.
Reason: The matrix representation of T is invertible, so T is both injective (nullity is zero) and surjective (full rank).

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V, let $P = P_{\mathcal{C}\leftarrow\mathcal{B}}$, and let $T : V \to V$ be a linear transformation. If A is the matrix of T with respect to $(\mathcal{B},\mathcal{B})$, then what is the matrix of T with respect to $(\mathcal{C},\mathcal{C})$?

A. PAP.

B. PAP^{-1}.

C. $P^{-1}AP$.

D. $P^{-1}AP^{-1}$.

E. None of these.

Answer: **B**.
Reason:

$$[T]_{\mathcal{C}}^{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}} = PAP^{-1}.$$

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V, let $T : V \to V$ be a linear transformation, and let A, A' be the matrices of T with respect to $(\mathcal{B},\mathcal{B})$, and $(\mathcal{C},\mathcal{C})$, respectively. Which of the following statements best describes common properties of A and A'?

A. $\det(A) = \det(A')$.

B. $\text{nullity}(A) = \text{nullity}(A')$.

C. $\text{rank}(A) = \text{rank}(A')$.

D. All of the above.

E. None of these.

Answer: **D**.
Reason: The matrices A and A' are similar, as in the last multiple choice problem. That is, $A' = PAP^{-1}$ for a change of basis matrix P. It follows that the rank and nullity of A are unchanged and the determinant satisfies $\det(A') = \det(P)\det(A)\det(P)^{-1} = \det(A)$.

Problem 7. (15 pts.) Define the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 2y - z \\ x + 3y \\ 2z \end{bmatrix}.$$

(a) Find the matrix representation of T with respect to the standard basis of $\mathbb{R}^3$.

Solution: We observe that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$
$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix},$$
$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

It follows that the matrix representation of T with respect to the standard basis is given by

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We can confirm our answer by checking that

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 2y - z \\ x + 3y \\ 2z \end{bmatrix}.$$

Problem 8. (20 pts.) Define the map

$$\begin{aligned} T : P_2 &\to P_2 \\ f(x) &\mapsto f(x) - xf'(x). \end{aligned}$$

(a) Show that T is linear.

Solution: Suppose c_1, c_2 are any constants and f_1, f_2 are any polynomials in P_2. Then

$$\begin{aligned} T\big((c_1f_1 + c_2f_2)(x)\big) &= (c_1f_1 + c_2f_2)(x) - x(c_1f_1 + c_2f_2)'(x) \\ &= c_1f_1(x) + c_2f_2(x) - c_1x1f_1'(x) - c_2xf_2'(x) \\ &= c_1\big(f_1(x) - xf_1'(x)\big) + c_2\big(f_2(x) - xf_2'(x)\big) \\ &= c_1T(f_1(x)) + c_2T(f_2(x)). \end{aligned}$$

As the c_i and f_i were arbitrary, this shows T is linear.

(b) Find a basis for the image of T.

Solution: On the standard basis $\{1, x, x^2\}$ of P_2, the transformation T acts by

$$\begin{aligned} T(1) &= 1 - 0 = 1, \\ T(x) &= x - x = 0, \\ T(x^2) &= x^2 - 2x^2 = -x^2. \end{aligned}$$

This means the matrix representation of T with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The image of T expressed in coordinates via the standard basis corresponds exactly the column space of A, which we see is spanned by $(1, 0, 0)$ and $(0, 0, -1)$ (since multiplying the last row by -1 would put A in row echelon form). This means the image of T has $\{1, -x^2\}$ or equivalently $\{1, x^2\}$ as a basis.

(continued →)

(c) Show that T is or is not injective (into) (one-to-one).

Solution: By the previous part, we have $T(x) = 0$. This means T is not injective because T has nontrivial kernel.

Or more directly, we could just note that both x and 0 map to 0, contradicting the definition of injectivity.

(d) Show that T is or is not surjective (onto).

Solution: By the previous part, the image has basis $\{1, x^2\}$. In particular, the rank of the map T (i.e. the rank of its matrix representation) is 2, which is less than the dimension of P_2. It follows that T is not surjective.

Or more directly, we could argue that x does not lie in the image of T since x is linearly independent of the basis of the image $\{1, x^2\}$.

Problem 9. (20 pts.) Let M_2 be the space of all 2×2 matrices with real entries, and define the map

$$\begin{aligned} T : M_2 &\to M_2 \\ B &\mapsto B + B^\top. \end{aligned}$$

(a) Show that T is linear.

Solution: Then

$$\begin{aligned} T\big(c_1B_1 + c_2B_2\big) &= (c_1B_1 + c_2B_2) + (c_1B_1 + c_2B_2)^\top \\ &= c_1B_1 + c_2B_2 + c_1B_1^\top + c_2B_2^\top \\ &= c_1(B_1 + B_1^\top) + c_2(B_2 + B_2)^\top \\ &= c_1T(B_1) + c_2T(B_2). \end{aligned}$$

As the c_i and B_i were arbitrary, this shows T is linear.

(b) Find the kernel of T and its dimension.

Solution: On the standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_2, the transformation T acts by

$$\begin{aligned} T(E_{11}) &= E_{11} + E_{11}^\top = 2E_{11}, \\ T(E_{12}) &= E_{12} + E_{12}^\top = E_{12} + E_{21}, \\ T(E_{21}) &= E_{21} + E_{12}^\top = E_{12} + E_{21}, \\ T(E_{22}) &= E_{22} + E_{22}^\top = 2E_{22}. \end{aligned}$$

This means the matrix representation of T with respect to the standard basis is

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is 3, so the nullity of A must be 1. This means the kernel of T must be <u>dimension one</u>. By the action of T above on basis elements, it's easy to see that $T(E_{12} - E_{21}) = 0$. This means

$$\operatorname{Ker}(T) = \operatorname{span}\{E_{12} - E_{21}\} = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

(continued $\to$)

(c) Find the range of T and its dimension.

Solution: From the matrix representation above, we see the column space of A has basis $(1,0,0,0)$, $(0,1,1,0)$, and $(0,0,0,1)$. This corresponds to the range (or image) of T having the basis $E_{11}, E_{12}+E_{21}, E_{22}$. Therefore,

$$\mathrm{Rng}(T) = \mathrm{span}\{E_{11}, E_{12}+E_{21}, E_{22}\},$$

which is precisely the set of symmetric 2×2 matrices. The range of T has dimension three.

(d) State an equation which relates $\dim(\ker(T))$ and $\dim(\mathrm{range}(T))$ to $\dim(M_2)$.

Solution: The generalized rank-nullity theorem states that

$$\mathrm{Ker}(T) + \mathrm{Rng}(T) = \dim(M_2).$$

In this problem, this equation is $1+3=4$.

Problem 10. (15 pts.) Define the three-dimensional space

$$V = \text{span}\{\sin^2(x),\, \cos^2(x),\, \sin(x)\cos(x)\}.$$

(a) Show that the derivatives of $\sin^2(x)$, $\cos^2(x)$, $\sin(x)\cos(x)$ all belong to V.

Solution: We compute that

$$\begin{aligned} \frac{d}{dx}\sin^2(x) &= 2\sin(x)\cos(x), \\ \frac{d}{dx}\cos^2(x) &= -2\sin(x)\cos(x), \\ \frac{d}{dx}\sin(x)\cos(x) &= \sin^2(x) - \cos^2(x). \end{aligned}$$

Each right hand side lies in V so the derivatives of the spanning functions also lie in V.

(b) By part (a), and since differention is linear, we have a linear map $\frac{d}{dx} : V \to V$. Find the matrix of $\frac{d}{dx}$ with respect to the basis $\{\sin^2(x),\, \cos^2(x),\, \sin(x)\cos(x)\}$.

Solution: Let $B = \{\sin^2(x),\, \cos^2(x),\, \sin(x)\cos(x)\}$. With respect to B, our formulas from part (a) imply

$$\left[\frac{d}{dx}\sin^2(x)\right]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \left[\frac{d}{dx}\cos^2(x)\right]_B = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad \left[\frac{d}{dx}\sin(x)\cos(x)\right]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

We conclude the matrix representation of $\frac{d}{dx}$ with respect to basis B is given by

$$\left[\frac{d}{dx}\right]_B^B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}.$$

Problem 11. (20 pts.) Define the linear map

$$T : P_2 \to P_2$$
$$p(x) \mapsto (x+2)p'(x)$$

(a) Find the matrix A of T with respect to the natural basis $\mathcal{B} = \{1, x, x^2\}$ of P_2.

Solution: We compute the action of T on the basis $\mathcal{B}$ is given by

$$\begin{aligned} T(1) &= (x+2)\cdot 0 = 0, \\ T(x) &= (x+2)\cdot 1 = 2 + x \\ T(x^2) &= (x+2)\cdot(2x) = 4x + 2x^2. \end{aligned}$$

This means that images of T on the natural basis have coordinate representations given by

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(x)]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad [T(x^2)]_B = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}.$$

The matrix A is therefore given by

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) Find the matrix M of T with respect to the basis $\mathcal{C} = \{1, x+2, (x+2)^2\}$ of P_2.

Solution: We use our computations above, but try to express the actions of T is the new basis:

$$\begin{aligned} T(1) &= (x+2)\cdot 0 = 0, \\ T(x+2) &= (x+2)\cdot 1 = (x+2) \\ T((x+2)^2) &= (x+2)\cdot(2(x+2)) = 2(x+2)^2 \end{aligned}$$

This means that images of T on the new basis have coordinate representations given by

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(x)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [T(x^2)]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

The matrix A is therefore given by

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(continued →)

(c) Find an invertible matrix Q such that A and M are related by $M = Q^{-1}AQ$.

Solution: Since $A = [T]_{\mathcal{B}}^{\mathcal{B}}$ and $M = [T]_{\mathcal{C}}^{\mathcal{C}}$ and in general

$$[T]_{\mathcal{C}}^{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}},$$

we should have $Q = P_{\mathcal{B}\leftarrow\mathcal{C}}$. To find Q, we express the basis elements of $\mathcal{C}$ in terms of the basis elements of $\mathcal{B}$. For this, we have

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[x+2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

$$[(x+2)^2]_{\mathcal{B}} = [x^2 + 4x + 4]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}.$$

Therefore

$$Q = P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can verify out answer by computing that

$$Q^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

and then checking that

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = Q^{-1}AQ.$$