MATH 205 PRACTICE EXAM ON NULLSPACES, COLUMN SPACES, AND ROW SPACES.

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 3×4 matrix. What can we say about the dimensions of its row space and column space?

A. $\dim(\text{rowspace}(A)) < \dim(\text{colspace}(A))$.

B. $\dim(\text{rowspace}(A)) = \dim(\text{colspace}(A))$.

C. $\dim(\text{rowspace}(A)) \neq \dim(\text{colspace}(A))$.

D. $\dim(\text{rowspace}(A)) > \dim(\text{colspace}(A))$.

E. None of these.

Answer: **B**.
Reason: Both the dimension of the column space and the dimension of the row space are equal to the number of pivot ones in an echelon form of A (also called the rank of A). So they are always equal.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an invertible 3×3 matrix. What is the nullity of A?

A. 0.

B. 1.

C. 2.

D. 3.

E. None of these.

Answer: **A**.
Reason: The nullity is the dimension of the nullspace. Since A is invertible, the only vector that A sends to $\mathbf{0}$ is the $\mathbf{0}$ vectors itself. Therefore the nullity is zero.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 4×4 matrix, let $\mathbf{b}$ be a vector in $\mathbb{R}^4$, and assume that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What is the rank of A?

A. 0.

B. 2.

C. 4.

D. 16.

E. None of these.
Answer: **C**.
Reason: We think of A as a linear map $A : \mathbb{R}^4 \to \mathbb{R}^4$. By the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = 4$. If the nullity were nonzero, then $A\mathbf{x} = \mathbf{b}$ would not have a unique solution (recall nullity is the number of free variables in a row echelon form of A). So the nullity must be zero and therefore, by rank-nullity, the rank of A is four.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 5×5 matrix with rank 4, and let $\mathbf{b}$ be a vector in $\mathbb{R}^5$. How many solutions does the equation $A\mathbf{x} = \mathbf{b}$ have?

A. 0.

B. 1.

C. Infinitely many.

D. It depends if $\mathbf{b}$ belongs to the column space of A or not.

E. None of these.

Answer: **D**.
Reason: We think of A as a linear map $A : \mathbb{R}^5 \to \mathbb{R}^5$. By the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = 5$. Since the rank of A is only 4, the nullity of A must be 1. Now there are two possibilities. First, it could be that b does not lie in the image of A (i.e. the column space of A). In this case, $A\mathbf{x} = \mathbf{b}$ has no solutions. On the other hand, if $\mathbf{b}$ does lie in the column space of A, then the row-echelon form of $[A|\mathbf{b}]$ will have free variables and so $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

Problem 5. (15 pts.) Let A be a 3×6 matrix,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \end{bmatrix},$$

and assume that performing row operations on A yields the matrix

$$B = \begin{bmatrix} 1 & 3 & 0 & 2 & 6 & -1 \\ 0 & 0 & 1 & 5 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the row space of A.

Solution: Since B is a row-echelon form of A, the rows of B form a basis for the row space of A. That is:

$$\text{rowspace}(A) = \text{span}\Big\{(1,3,0,2,6,-1),(0,0,1,5,-2,-3)\Big\}.$$

(b) Find a basis for the column space of A.

Solution: Since B is a row-echelon form of A, the columns of B that contain pivot (or leading) ones form a basis for the column space of A. These columns are the first and third columns. That is:

$$\text{colspace}(A) = \text{span}\Big\{(a_{11},a_{21},a_{31}),(a_{13},a_{23},a_{33})\Big\}.$$

Notice how the dimension of the row space and the dimension of the column space are equal and equal the rank of A.

(c) Find the dimension of the nullspace of A.

Solution: We think of A as a linear map $A : \mathbb{R}^6 \to \mathbb{R}^3$. By rank-nullity, we have $\text{rank}(A) + \text{nullity}(A) = 6$. Since the rank of A is 2, the nullity of A (i.e. dimension of the nullspace) must be 4.

Problem 6. (20 pts.) Define the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 1 & -6 & 7 \end{bmatrix}.$$

(a) Find a basis of the row space of A.

Solution: We first row-reduce the matrix A.

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 1 & -6 & 7 \end{bmatrix} \xrightarrow{A_{12}(-2), A_{13}(-1)} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{bmatrix} \xrightarrow{A_{23}(1), M_2(\frac{1}{5})} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of the echelon form of A form a basis for the row space of A. Therefore a basis for the row space of A is

$$\Big\{(1, -1, 3), (0, 1, -4/5)\Big\}.$$

(continued →)

(b) Find a basis of the column space of A.

Solution: A basis for the column space of A is given by the columns of A that correspond to columns in a row echelon form of A that contain leading ones. By the previous problem, we found that the first and second columns of a row echelon form of A contain leading ones. Therefore a basis for the column space of A is given by

$$\Big\{(1,2,1),(-1,3,-6)\Big\}.$$

(c) Find a basis of the nullspace of A.

Solution: Since the rank of A is 2, the nullity of A must be 1 (by rank-nullity). To find the solution space of $A\mathbf{x} = \mathbf{0}$, we would row reduce the augmented matrix $[A|\mathbf{0}]$. By our work in (a) above, this gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -4/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking $x_3 = t$, we find $x_2 = \frac{4/5}{t}$ and $x_1 = -\frac{11}{5}t$. Taking $t = 5$, we find a basis for the nullspace is

$$\Big\{(-11,4,5)\Big\}.$$

Problem 7. (15 pts.) Let A be a 3×4 matrix, let $\mathbf{x}$ be a vector of 4 variables, and let $\mathbf{b}$ be a vector in $\mathbb{R}^3$. Suppose that one solution of the linear system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix},$$

and that a basis of the nullspace of A is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \right\}.$$

Describe all solutions of the linear system $A\mathbf{x} = \mathbf{b}$.

Solution: In general, all solutions of a problem $A\mathbf{x} = \mathbf{b}$ are of the form $\mathbf{x}_p + c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$ where $\mathbf{x}_p$ is a particular solution, $\mathbf{x}_1, \ldots \mathbf{x}_k$ is a basis for the nullspace of A, and $c_1, \ldots, c_k$ are undetermined coefficients.

In the present problem, this means all solutions of $A\mathbf{x} = \mathbf{b}$ are of the form

$$\begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}.$$