MATH 205 PRACTICE EXAM ON COORDINATE VECTORS AND CHANGE OF BASIS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let elements $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of a vector space V form a basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, and let $\mathbf{v} \in V$ satisfy $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for real numbers c_1, c_2, c_3. What is the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$?

A. $c_1c_2c_3$.

B. $c_1 + c_2 + c_3$.

C. $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

D. $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

E. Other.

Answer: **C**.
Reason: The coordinate vector expression is an element of $\mathbb{R}^3$, who entries correspond to the scalar coefficients of each basis element of $\mathcal{B}$ when expressing $\mathbf{v}$ as a linear combination of them.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let elements $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of a vector space V form a basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. What is the coordinate vector $[\mathbf{u}_1]_{\mathcal{B}}$ of $\mathbf{u}_1$ with respect to $\mathcal{B}$?

A. 0.

B. 1.

C. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

D. $\mathbf{u}_1$.

E. Other.

Answer: **C**.
Reason: As in the last problem, consider the coordinate expression of $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, but take $c_1 = 1$, $c_2 = c_3 = 0$.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

What element of $\mathbb{R}^3$ has coordinate vector

$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

with respect to the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}?$$

A. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

B. $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

C. $\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$.

D. $\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$.

E. Other.

Answer: **C**.
Reason: Observe that

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}.$$

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

The polynomial $2 - 3x - 7x^2$ has coordinate vector

$$\begin{bmatrix} 2 \\ -3 \\ -7 \end{bmatrix}$$

with respect to what basis of P_2?

A. $\{1, x, x^2\}$.

B. $\{2, -3x, -7x^2\}$.

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

D. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} \right\}$.

E. Other.

Answer: **A**.
Reason: Observe that

$$2(1) - 3(x) - 7(x^2) = 2 - 3x - 7x^2.$$

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases of a vector space V. How are the change-of-basis matrices $P_{\mathcal{B}\leftarrow\mathcal{C}}$ and $P_{\mathcal{C}\leftarrow\mathcal{B}}$ related?

A. $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

B. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^\top$.

C. $P_{\mathcal{C}\leftarrow\mathcal{B}} = (P_{\mathcal{B}\leftarrow\mathcal{C}})^{-1}$.

D. $P_{\mathcal{C}\leftarrow\mathcal{B}} = ((P_{\mathcal{B}\leftarrow\mathcal{C}})^\top)^{-1}$.

E. None of these.

Answer: **C**.
Reason: The matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is defined on coordinate representations: $[\mathbf{v}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_\mathcal{B}$. Hence, $[\mathbf{v}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{v}]_\mathcal{C}$ for any $\mathbf{v}$. It follows that $P_{\mathcal{C}\leftarrow\mathcal{B}}, P_{\mathcal{B}\leftarrow\mathcal{C}}$ are inverses.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be bases of a vector space V. Which of the following describes the change-of-basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$?

A. Its jth column is the coordinate vector of w_j with respect to $\mathcal{B}$.

B. Its jth column is the coordinate vector of v_j with respect to $\mathcal{C}$.

C. Its jth column is the coordinate vector of v_j with respect to $\mathcal{B}$.

D. Its jth column is the coordinate vector of w_j with respect to $\mathcal{C}$.

E. None of these.

Answer: **A**.
Reason: The jth column of $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is the coordinate expression for the jth basis element of $\mathcal{C}$ in the basis $\mathcal{B}$. Therefore, the jth column should be the expression of the jth coordinate vector $\mathbf{w}_j$ with respect to $\mathcal{B}$.

Problem 7. (20 pts.) Define the two bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} \right\}$$

of the space of symmetric 2×2 matrices $V = \{A \in M_2 \mid A^{\top} = A\}$.

(a) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

Solution: We express the elements of the basis $\mathcal{C}$ in terms of the basis $\mathcal{B}$. To that end, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall that the columns of $P_{\mathcal{B}\leftarrow\mathcal{C}}$ are precisely the coefficients that appear above, we deduce that

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution: $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the inverse of the matrix we found above. So we do Gauss-Jordan elimination:

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 14 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \to R_1 + 3R_2,$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 14 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_2 \to R_2 - 4R_3,$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -14 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \to R_1 - 14R_3.$$

Therefore

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 3 & -14 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Find the coordinate vector of the matrix

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix}$$

with respect to $\mathcal{B}$.

Solution: With respect to $\mathcal{B}$, it is straightforward to see that

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix} = 8 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the coordinate expression with respect to $\mathcal{B}$ of this matrix is

$$\begin{bmatrix} 8 \\ -1 \\ 6 \end{bmatrix}.$$

(d) Find the coordinate vector of the matrix

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix}$$

with respect to $\mathcal{C}$.

Solution: We use our change of basis matrix $P_{mathcalC \leftarrow \mathcal{B}}$ to find the coordinate expression with respect to $\mathcal{C}$ is:

$$\begin{bmatrix} 1 & 3 & -14 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 8-3-84 \\ -1-24 \\ 6 \end{bmatrix} = \begin{bmatrix} -79 \\ -25 \\ 6 \end{bmatrix}.$$

In other words,

$$\begin{bmatrix} 8 & -1 \\ -1 & 6 \end{bmatrix} = -79 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 25 \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}.$$

Problem 8. (15 pts.) Define the three bases

$$\mathcal{B} = \{1, \quad x, \quad x^2\},$$
$$\mathcal{C} = \{1, \quad 1+x, \quad 1+x+x^2\},$$
$$\mathcal{C}' = \{1+x+x^2, \quad x+x^2, \quad x^2\}$$

of P_2.

(a) Find the change of basis matrix $P_{\mathcal{C}' \leftarrow \mathcal{C}}$. You may use the facts that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

and $P_{\mathcal{C}' \leftarrow \mathcal{C}} = P_{\mathcal{C}' \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{C}}$.

Solution: We first observe that

$$P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_{\mathcal{B} \leftarrow \mathcal{C}'} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now recall $P_{\mathcal{C}' \leftarrow \mathcal{B}} = P^{-1}_{\mathcal{B} \leftarrow \mathcal{C}'}$. Therefore

$$\begin{aligned} P_{\mathcal{C}' \leftarrow \mathcal{C}} = P^{-1}_{\mathcal{B} \leftarrow \mathcal{C}'} P_{\mathcal{B} \leftarrow \mathcal{C}} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned}$$

(b) If polynomial $p(x)$ has coordinate vector

$$\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$$

with respect to $\mathcal{C}$, then what is its coordinate vector with respect to $\mathcal{C}'$?

Solution: We multiply the vector by $P_{\mathcal{C}' \leftarrow \mathcal{C}}$ to obtain the desired coordinate vector:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \\ -8 \end{bmatrix}.$$

Problem 9. (20 pts.) Define the basis

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$. If $\mathbf{v}$ is the point

$$\begin{bmatrix} 21 \\ 7 \\ 15 \end{bmatrix}$$

in $\mathbb{R}^3$ (expressed in terms of the natural basis of unit vectors), then what is the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{C}$?

Solution: Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the natural basis of unit vectors so that

$$\mathbf{v} = 21\mathbf{e}_1 + 7\mathbf{e}_2 + 15\mathbf{e}_3.$$

Observe that

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

We compute its inverse, $P_{\mathcal{C}\leftarrow\mathcal{B}}$, through row-reduction ($R_3 \to R_3 - R_1$ followed by $R_1 \to R_1 - 3R_3$)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right],$$

obtaining

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Now we apply $P_{\mathcal{C}\leftarrow\mathcal{B}}$ to the $\mathcal{B}$-coordinate expression of $\mathbf{v}$ to get its coordinate with respect to $\mathcal{C}$:

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 7 \\ 15 \end{bmatrix} = \begin{bmatrix} 27 \\ 7 \\ -6 \end{bmatrix}.$$

Indeed, we can verify that

$$\begin{bmatrix} 21 \\ 7 \\ 15 \end{bmatrix} = 27 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$