

MATH 205 PRACTICE EXAM ON VECTOR SPACES AND BASES

Name:

Section:

Instructions:

- (1) Books, notes, and calculators are not permitted.
- (2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
- (3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (18 pts.) For each of the following sets, write a check mark $\checkmark$ after it if it is a vector space; write an X after it if it is not a vector space. Let P_k denote the set of polynomials of degree at most k in a single variable x , having real coefficients.

There will be no partial credit. You do not need to show your work.

(i) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \text{ are even integers} \right\}.$

(ii) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$

(iii) $\{p(x) \in P_3 \mid \deg(p(x)) = 3\}.$

(iv) $\{p(x) \in P_3 \mid p(0) = 0\}.$

(v) $\{p(x) \in P_3 \mid p(1) = 0\}.$

(vi) $\{p(x) \in P_3 \mid p(0) = 1\}.$

Problem 2. (18 pts.) For each of the following sets, write a check mark ✓ after it if it is a vector space; write an X after it if it is not a vector space. Let M_n denote the set of $k \times k$ matrices having real entries.

There will be no partial credit. You do not need to show your work.

(i) $\{A \in M_2 \mid a_{1,1} + a_{2,2} = 0\}.$

(ii) $\{A \in M_2 \mid \det(A) \neq 0\}.$

(iii) $\{A \in M_k \mid A^\top = A\}.$

(iv) $\{A \in M_k \mid A^\top = -A\}.$

(v) $\left\{ \begin{bmatrix} a-b \\ a \\ a+b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$

(vi) $\left\{ \begin{bmatrix} a-1 \\ a \\ a+1 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the three sets

$$S_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\},$$

$$S_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \right\},$$

$$S_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

Which of the sets is linearly independent?

- A. S_1 only.
- B. S_2 only.
- C. S_1 and S_3 only.
- D. S_2 and S_3 only.
- E. Other.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Adjoining

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

to the sets from the previous problem, we obtain the three sets

$$T_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$
$$T_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$
$$T_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Which of these sets spans $\mathbb{R}^3$?

- A. T_1 only.
- B. T_2 only.
- C. T_1 and T_3 only.
- D. T_2 and T_3 only.
- E. Other.

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \{1, \quad 1 + x, \quad 1 + x + x^2, \quad x + x^2, \quad x^2\}$$

of P_2 . Which of the following best describes S ?

- A. It is a basis of P_2 .
- B. It spans P_2 but is not linearly independent.
- C. It is linearly independent but does not span P_2 .
- D. It neither spans P_2 nor is linearly independent.
- E. Other.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

of M_2 . Which of the following best describes S ?

- A. It is a basis of M_2 .
- B. It spans M_2 but is not linearly independent.
- C. It is linearly independent but does not span M_2 .
- D. It neither spans M_2 nor is linearly independent.
- E. Other.

Problem 7. (15 pts.) For each of the sets below, show that it is linearly independent, or find a nontrivial linear combination of the elements which equals 0.

(a)

$$\left\{ \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \right\}.$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c)

$$\{1, \quad 1+x, \quad (1+x)^2\}.$$

(d)

$$\{1+x, \quad 1-x, \quad 1+x^2, \quad 1-x^2\}.$$

Problem 8. (25 pts.) For each of the following sets, show that it is or is not closed under addition and scalar multiplication. Then give one natural basis (or state that it is not a vector space), and find its dimension as a real vector space (or answer “DNA” if the set is not a vector space).

(a) The set $P_3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$.

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

(continued $\rightarrow$)

(b) The subset V_1 of P_3 consisting of polynomials whose coefficients sum to 0,

$$V_1 = \{a + bx + cx^2 + dx^3 \in P_3 \mid a + b + c + d = 0\}.$$

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

(continued $\rightarrow$)

(c) The subset V_2 of V_1 consisting of polynomials having constant term equal to the coefficient of x^2 and coefficient of x equal to the coefficient of x^3 ,

$$V_2 = \{a + bx + cx^2 + dx^3 \in P_3 \mid a + b + c + d = 0, a = c, b = d\}.$$

(i) Closure under addition:

(ii) Closure under scalar multiplication:

(iii) One natural basis:

(iv) Dimension:

Problem 9. (10 pts.) Consider the linearly independent set R and the spanning set T of $\mathbb{R}^3$ given by

$$R = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \quad T = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} \right\}.$$

Find a basis S of $\mathbb{R}^3$ satisfying $R \subset S \subset T$. Briefly justify your answer.

Problem 10. (10 pts.) Consider the space of continuous functions spanned by $\sin(x)$, $\sin(2x)$, and $\sin(3x)$,

$$V = \text{span}(\sin(x), \sin(2x), \sin(3x)).$$

Use the Wronskian and a strategically chosen value of x to decide if $\sin(x)$, $\sin(2x)$, $\sin(3x)$ are linearly independent functions of x and therefore form a basis of V .