MATH 205 PRACTICE EXAM ON VECTOR SPACES AND BASES

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (18 pts.) For each of the following sets, write a check mark ✓ after it if it is a vector space; write an X after it if it is not a vector space. Let P_k denote the set of polynomials of degree at most k in a single variable x, having real coefficients.

There will be no partial credit. You do not need to show your work.

(i) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a, b \text{ are even integers} \right\}.$ **X**

Reason: This set isn't closed under multiplication by scalars (of real numbers).

(ii) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$ ✓

Reason: This is a standard vector space $\mathbb{R}^2$.

(iii) $\{p(x) \in P_3 \,|\, \deg(p(x)) = 3\}.$ **X**

Reason: This set doesn't contain the zero element (the degree of the zero polynomial is 0).

(iv) $\{p(x) \in P_3 \,|\, p(0) = 0\}.$ ✓

Reason: This is the kernel of the evaluation map $P_3 \to \mathbb{R}$ sending $p(x) \to p(0)$. We will see the kernel of a linear map between vector spaces (an evaluation map is a linear map) is always a vector space. Alternatively, you can check that the set contains 0 and the condition is $p(0) = 0$ is preserved under addition and scalar multiplication.

(v) $\{p(x) \in P_3 \,|\, p(1) = 0\}.$ ✓

Reason: Same as previous case.

(vi) $\{p(x) \in P_3 \,|\, p(0) = 1\}.$ **X**

Reason: The set does not contain the zero polynomial.

Problem 2. (18 pts.) For each of the following sets, write a check mark ✓ after it if it is a vector space; write an X after it if it is not a vector space. Let M_n denote the set of $k \times k$ matrices having real entries.

There will be no partial credit. You do not need to show your work.

(i) $\{A \in M_2 \,|\, a_{1,1} + a_{2,2} = 0\}$. ✓

Reason: This is the set of matrices with $\mathrm{tr}(A) = 0$. Since $\mathrm{tr} : M_n \to \mathbb{R}$ is a linear map, this set is the kernel of a linear map, hence a vector space.

(ii) $\{A \in M_2 \,|\, \det(A) \neq 0\}$. **X**

Reason: Doesn't contain the zero matrix.

(iii) $\left\{A \in M_k \,|\, A^\top = A\right\}$. ✓

Reason: This is the set of symmetric matrices, which contains the zero matrix and is preserved under addition and scalar multiplication. (It is also the kernel of the map $A \mapsto A - A^T$, which is linear).

(iv) $\left\{A \in M_k \,|\, A^\top = -A\right\}$. ✓

Reason: Same as previous case. Reason: This is the set of anti-symmetric matrices, which contains the zero matrix and is preserved under addition and scalar multiplication. (It is also the kernel of the map $A \mapsto A + A^T$, which is linear).

(v) $\left\{ \begin{bmatrix} a-b \\ a \\ a+b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$. ✓

Reason: This is the span of the vectors $(1,1,1)$ and $(-1,0,1)$ based at the origin $(0,0,0)$. Spans of vectors (based at the origin) are vector spaces.

(vi) $\left\{ \begin{bmatrix} a-1 \\ a \\ a+1 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$. **X**

Reason: This set doesn't contain $(0,0,0)$.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the three sets

$$S_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\},$$
$$S_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \right\},$$
$$S_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

Which of the sets is linearly independent?

A. S_1 only.

B. S_2 only.

C. S_1 and S_3 only.

D. S_2 and S_3 only.

E. Other.

Answer: **A**.
Reason: The second vector of S_2 is a multiple of the first. The third vector of S_3 is the sum of the first two.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Adjoining

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

to the sets from the previous problem, we obtain the three sets

$$T_1 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$T_2 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$T_3 = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Which of these sets spans $\mathbb{R}^3$?

A. T_1 only.

B. T_2 only.

C. T_1 and T_3 only.

D. T_2 and T_3 only.

E. Other.

Answer: **C**.
Reason: Since S_1 had 3 linearly independent vectors, it spanned $\mathbb{R}^3$. Since T_1 contains S_1 it also spans $\mathbb{R}^3$. In T_2, the second two vectors are multiples of the first. So T_2 contains at most 2 linearly independent vectors, therefore cannot span $\mathbb{R}^3$. In T_3, the first, second, and fourth vectors are independent. Indeed

$$\det \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = 3 \neq 0.$$

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \{1, \quad 1+x, \quad 1+x+x^2, \quad x+x^2, \quad x^2\}$$

of P_2. Which of the following best describes S?

A. It is a basis of P_2.

B. It spans P_2 but is not linearly independent.

C. It is linearly independent but does not span P_2.

D. It neither spans P_2 nor is linearly independent.

E. Other.

Answer: **B**.
Reason: The dimension of P_2 is 3 (the standard basis is $1, x, x^2$). So S cannot be a basis (it contains too many elements to be linearly independent), however we can see it spans P_2 because

$$a + bx + cx^2 = (a-b) \cdot 1 + b \cdot (1+x) + c \cdot x^2.$$

So we've expressed an arbitrary element of P_2 as a linear combination of $1, 1+x, x^2$, which are elements of S.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the subset

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

of M_2. Which of the following best describes S?

A. It is a basis of M_2.

B. It spans M_2 but is not linearly independent.

C. It is linearly independent but does not span M_2.

D. It neither spans M_2 nor is linearly independent.

E. Other.

Answer: **C**.
Reason: The dimension of M_2 is 4. The standard basis is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So S cannot be a basis (it contains too few elements to span M_2), however we can see the elements of S are linearly independent since if for some (c_1, c_2, c_3) we have

$$0 = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

Then evidently $c_1 = c_2 = c_3 = 0$.

Problem 7. (15 pts.) For each of the sets below, show that it is linearly independent, or find a nontrivial linear combination of the elements which equals 0.

(a)

$$\left\{ \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \right\}.$$

Solution: Not linearly independent. Indeed:

$$\frac{3}{2} \cdot \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} = 0.$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution: Not linearly independent. Add them all to get

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c)

$$\left\{ 1, \quad 1+x, \quad (1+x)^2 \right\}.$$

Solution: Linearly independent. If for all x,

$$0 = c_1 \cdot 1 + c_2 \cdot (1+x) + c_3 \cdot (1+x)^2$$

then taking $x = -1$ implies $c_1 = 0$. Then $(1+x)(c_2 + c_3(1+x)) = 0$ for all x, hence $c_2 + c_3(1+x) = 0$ for all x. Taking $x = -1$ again, we get $c_2 = 0$. Hence finally $c_3 = 0$. So there is no linear relation between the elements. (d)

$$\left\{ 1+x, \quad 1-x, \quad 1+x^2, \quad 1-x^2 \right\}.$$

Solution: Not linearly independent. Indeed,

$$(1+x) + (1-x) - (1+x^2) - (1-x^2) = 0.$$

Problem 8. (25 pts.) For each of the following sets, show that it is or is not closed under addition and scalar multiplication. Then give one natural basis (or state that it is not a vector space), and find its dimension as a real vector space (or answer "DNA" if the set is not a vector space).

(a) The set $P_3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$.

(i) Closure under addition:
Solution: Given two cubic polynomials $p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and $p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3$ with real-valued coefficients, evidently the addition of them yields

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3$$

which is a cubic polynomial with real-valued coefficients.

(ii) Closure under scalar multiplication:
Solution: Given a cubic polynomials $p(x) = a + bx + cx^2 + dx^3$ and with real-valued coefficients and a real number s, the scalar multiplication is

$$s(p(x)) = (sa) + (sb)x + (sc)x^2 + (sd)x^3$$

which is again a cubic polynomial with real-valued coefficients.

(iii) One natural basis:
Solution: A natural basis is simply $\{1, x, x^2, x^3\}$. Indeed, the set P_3 is expressed as all linear combinations of these monomials. There are no linear relationships between polynomials of different degrees.

(iv) Dimension:
Solution: The dimension of P_3 is 4 (the natural basis above consists of exactly four elements) .

(continued →)

(b) The subset V_1 of P_3 consisting of polynomials whose coefficients sum to 0,

$$V_1 = \{a + bx + cx^2 + dx^3 \in P_3 \,|\, a + b + c + d = 0\}.$$

(i) Closure under addition:

Solution: Consider two cubic polynomials $p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and $p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3$ with real-valued coefficients, such that $a_i + b_i + c_i + d_i = 0$ for $i = 1, 2$. Recall from above the addition of these polynomials is

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3.$$

We verify that

$$(a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2) = (a_1+b_1+c_1+d_1)+(a_2+b_2+c_2+d_2) = 0+0 = 0.$$

So the subset V_1 is closed under addition.

(ii) Closure under scalar multiplication:

Solution: Consider a cubic polynomials $p(x) = a + bx + cx^2 + dx^3$ and with real-valued coefficients such that $a+b+c+d = 0$ and a real number s. Recall from above that the scalar multiplication is

$$s(p(x)) = (sa) + (sb)x + (sc)x^2 + (sd)x^3.$$

We verify that

$$(sa) + (sb) + (sc) + (sd) = s(a + b + c + d) = s \cdot 0 = 0.$$

So the subset V_1 is closed under scalar multiplication.

(iii) One natural basis:

Solution: To ensure we have the required relation between the coefficients, we choose our basis so that scalar multiples automatically have coefficients that add to zero. Namely, we take for a basis $\{x - 1, x^2 - 1, x^3 - 1\}$. Indeed, for any element of V_1, we have

$$\begin{aligned} a + bx + cx^2 + dx^3 &= (a + b + c + d) + b(x - 1) + c(x^2 - 1) + d(x^3 - 1) \\ &= b(x - 1) + c(x^2 - 1) + d(x^3 - 1). \end{aligned}$$

Thus every element of V_1 can be expressed as a linear combination of our basis elements. Additionally, there are no linear relationships between polynomials of different degrees.

(iv) Dimension:

Solution: The dimension of V_1 is 3 (the natural basis above consists of exactly three elements) .

(continued $\rightarrow$)

(c) The subset V_2 of V_1 consisting of polynomials having constant term equal to the coefficient of x^2 and coefficient of x equal to the coefficient of x^3,

$$V_2 = \{a + bx + cx^2 + dx^3 \in P_3 \,|\, a + b + c + d = 0,\ a = c,\ b = d\}.$$

(i) Closure under addition:

Solution: Consider two cubic polynomials $p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and $p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3$, which lie in V_2. Recall from above the addition of these polynomials is

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3.$$

We have already seen above that $p_1(x)+p_2(x)$ has coefficients which sum to zero. We verify that

$$(a_1 + a_2) - (c_1 + c_2) = (a_1 - c_1) + (a_2 - c_2) = 0$$

and, by similarly identity, we find $(b_1 + b_2) = (d_1 + d_2)$. So the subset V_2 is closed under addition.

(ii) Closure under scalar multiplication:

Solution: Consider a cubic polynomials $p(x) = a + bx + cx^2 + dx^3$ which lies in V_2 a real number s. Recall from above that the scalar multiplication is

$$s(p(x)) = (sa) + (sb)x + (sc)x^2 + (sd)x^3.$$

We have already seen above that $s(p(x))$ has coefficients which sum to zero. We verify that

$$(sa) - (sc) = s(a - c) = s \cdot 0 = 0,$$

and, by a similar identity, $sb = sd$. So the subset V_2 is closed under scalar multiplication.

(iii) One natural basis:

Solution: Since V_2 is determined by 3 linear relations, we expect it's dimension to be $\dim P_3 - 3 = 1$. We can find a basis element by setting the coefficient $a = 1$ and using the relations to determine $c = 1$, $b = d = -1$. So a natural basis element is given by $\{1 - x + x^2 - x^3\}$. Indeed, every element of V_2 can be expressed as

$$\begin{aligned} a + bx + cx^2 + dx^2 &= (b + a)x + (c - a)x^2 + (d + a)x^3 + a(1 - x + x^2 - x^3) \\ &= a(1 - x + x^2 - x^3), \end{aligned}$$

using that $a + b = a - c = a + d = 0$ for elements of V_2.

(iv) Dimension:

Solution: As in the previous part, we see that dimension of V_2 is 1.

Problem 9. (10 pts.) Consider the linearly independent set R and the spanning set T of $\mathbb{R}^3$ given by

$$R = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \qquad T = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} \right\}.$$

Find a basis S of $\mathbb{R}^3$ satisfying $R \subset S \subset T$. Briefly justify your answer.

Solution: To make a basis S, we need to choose one element from T to add to R. Let us label the elements of R by $\mathbf{r}_1$ and $\mathbf{r}_2$ and the elements of T by $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_5$. Observe that $\mathbf{t}_1, \mathbf{t}_2$ are already in R, so adding them R won't give us a basis. On the other hand, $\mathbf{t}_3 = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{t}_5 = \mathbf{r}_1 + \mathbf{r}_2$, so these elements can already be made from linear combinations of the elements of R. Since we are told that T is a spanning set and we know the dimension of $\mathbb{R}^3$ is 3, adding the remaining element in T to R must give a basis. So we take

$$S = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}_4\}.$$

Problem 10. (10 pts.) Consider the space of continuous functions spanned by $\sin(x)$, $\sin(2x)$, and $\sin(3x)$,

$$V = \operatorname{span}(\sin(x), \sin(2x), \sin(3x)).$$

Use the Wronskian and a strategically chosen value of x to decide if $\sin(x)$, $\sin(2x)$, $\sin(3x)$ are linearly independent functions of x and therefore form a basis of V.

Solution: The Wronskian of the three functions is given by

$$W(x) := \det \begin{bmatrix} \sin(x) & \sin(2x) & \sin(3x) \\ \cos(x) & 2\cos(2x) & 3\cos(3x) \\ -\sin(x) & -4\sin(2x) & -9\sin(3x) \end{bmatrix}$$

Taking $x = \frac{\pi}{2}$, we have

$$W(\frac{\pi}{2}) := \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 9 \end{bmatrix} = 1 \cdot (-2) \cdot (9) - (-1) \cdot (-2) \cdot (-1) = -18 + 2 = -16.$$

Since the Wronskian doesn't vanish, the functions must be linearly independent.