MATH 205 PRACTICE EXAM ON THE DETERMINANT AND CRAMER'S RULE

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the invertible matrices

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 4 \\ 7 & 5 \end{bmatrix}.$$

Which of the inverses A^{-1}, B^{-1}, C^{-1} have entries which are all integers?

A. A^{-1} only.

B. A^{-1} and B^{-1} only.

C. A^{-1}, B^{-1} and C^{-1}.

D. B^{-1} only.

E. Other.

Answer: **B**.
Reason: Recall the inverse involves dividing the matrix entries by $\det A$ (see exercise 8 on pex3).

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be a 5×5 matrix having determinant 3. Which of the following is the determinant of $2A$?

A. 3.

B. 6.

C. 30.

D. 96.

E. None of these.

Answer: **D**.
Reason: A is $n \times n$ implies $\det(\lambda A) = \lambda^n \det(A)$.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Define the matrix
$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$
Which of the following is equal to $\det(A)$?

A. 0.

B. 3.

C. 12.

D. 81.

E. Other.

Answer: **D**.
Reason: The determinant of an upper (or lower) triangular matrix is the multiplication of diagonal terms.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. If $\det(A) = 2$, then how many solutions does the system $A\mathbf{x} = \mathbf{b}$ have?

A. 1.

B. 2.

C. $2n$.

D. 2^n.

E. None of these.

Answer: **A**.
Reason: A is invertible if and only if $\det(A) \neq 0$. If A invertible, the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Problem 5. (15 pts.) Let A and B be 3×3 matrices satisfying

$$\det(A) = 3, \qquad \det(B) = -4.$$

Let A' be the matrix obtained from A by multiplying its second row by 3, and by replacing the third row by (first row of A + third row of A).

Let B' be the matrix obtained from B by multiplying its first column by 4 and by swapping its second and third columns.

Compute the determinants of the following matrices.

Solution:

(a) $2A$.

$$\det(2A) = 2^3 \det(A) = 8 \cdot 3 = \boxed{24.}$$

(b) A^{-1}.

$$\det(A^{-1}) = \big(\det(A)\big)^{-1} = 3^{-1} = \boxed{\frac{1}{3}.}$$

(c) $A^{\intercal} B$.

$$\det(A^{\intercal} B) = \det(A^{\intercal})\det(B) = \det(A)\det(B) = 3 \cdot (-4) = \boxed{-12.}$$

(d) B^5.

$$\det(B^5) = \det(B)^5 = (-4)^5 = \boxed{-1024.}$$

(e) $(B^{-1}AB)^2$.

$$\det((B^{-1}AB)^2) = \det(B^{-1}AB)^2 = \det(B)^{-2}\det(A)^2\det(B)^2 = \det(A)^2 = \boxed{9.}$$

(f) A'

Multiplying a row multiplies the matrix by the factor; adding rows does not change the determinant:

$$\det(A') = 3\det(A) = \boxed{9.}$$

(g) B'.

Multiplying a column multiplies the matrix by the factor; swapping columns introduces a negative sign:

$$\det(B') = -4\det(B) = \boxed{16.}$$

Problem 6. (15 pts.) Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 1 & 1 & 0 & -2 \\ 6 & 0 & -1 & -2 \\ -3 & 1 & 3 & 2 \end{bmatrix}.$$

Show your work. **Solution:**

(a) $\det(A) =$

We first do row reduction: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$:

$$\begin{bmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & 7 \\ 0 & -13 & -12 \\ 0 & -6 & -10. \end{bmatrix}$$

Next we do Laplace expansion along the first column:

$$\det(A) = 2 \det \begin{bmatrix} -13 & -12 \\ -6 & -10 \end{bmatrix} = 2\big((-13)(-10) - (-12)(-6)\big) = \boxed{116.}$$

(a) $\det(B) =$

We first do the row reduction $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 6R_1$, and $R_4 \to R_4 - 3R_1$. Then we do $R_4 \to R_4 - R_2$ to the result to obtain:

$$\begin{bmatrix} -1 & 0 & 0 & 4 \\ 1 & 1 & 0 & -2 \\ 6 & 0 & -1 & -2 \\ -3 & 1 & 3 & 2 \end{bmatrix}. \to \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 22 \\ 0 & 1 & 3 & -10 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 22 \\ 0 & 0 & 3 & -12 \end{bmatrix}.$$

Next we do Laplace expansion along the first column to get

$$\det(B) = (-1) \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 22 \\ 0 & 3 & -12 \end{bmatrix} = -\det \begin{bmatrix} -1 & 22 \\ 3 & -12 \end{bmatrix} = -\big((-1)(-12) - (22)(3)\big) = \boxed{54.}$$

Problem 7. (15 pts.) Define the matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$$

Use Cramer's rule to solve the system

$$A\mathbf{x} = \mathbf{b}.$$

Show your work.

Solution: Cramer's rule says that the solution is given by $x_i = \det(A)^{-1}\det(A_i)$ where A_i is the matrix obtained by replacing the i-th column by the column vector $\mathbf{b}$. We compute that

$$\det(A) = 2\det\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} - \det\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = (2)(-3) - (1)(-2) + (1)(5) = 1.$$

Similarly

$$\begin{aligned} \det(A_1) &= \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \\ &= 4\det\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + 3\det\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = (4)(-3) - (2)(-2) + (3)(5) = 7, \end{aligned}$$

$$\begin{aligned} \det(A_2) &= \begin{bmatrix} 2 & 4 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \\ &= -\left(4\det\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\right) = -(4)(-1) + (2)(-1) - (3)(1) = -1, \end{aligned}$$

and

$$\begin{aligned} \det(A_2) &= \begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \\ &= 4\det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = (4)(2) - (2)(1) + (3)(-3) = -3. \end{aligned}$$

We conclude $x_1 = 7$, $x_2 = 1$ and $x_3 = -3$, or equivalently

$$\boxed{\mathbf{x} = \begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix}.}$$

Problem 8. (15 pts.) Use the adjoint method to invert the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 5 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Show your work.

Solution: The adjoint inverse formula asserts that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

where the adjugate matrix $\operatorname{adj}(A) = C^\top$ and C has entries (c_{ij}) given by $c_{ij} = (-1)^{i+j}\det(M_{ij})$ where M_{ij} is the matrix obtained by deleting the ith row and jth column from A. Procedding, we compute

$$c_{11} = \det \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} = 16, \qquad c_{12} = -\det \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = -1, \quad c_{13} = \det \begin{bmatrix} 0 & 5 \\ 1 & 1 \end{bmatrix} = -5,$$

$$c_{21} = -\det \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} = 4, \qquad c_{22} = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5, \qquad c_{23} = -\det \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = -3,$$

$$c_{31} = \det \begin{bmatrix} -1 & 1 \\ 5 & -1 \end{bmatrix} = -4, \quad c_{32} = -\det \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = 2, \qquad c_{33} = \det \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} = 10.$$

On the other hand,

$$\det(A) = 2\det \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} + \det \begin{bmatrix} -1 & 1 \\ 5 & -1 \end{bmatrix} = 32 + -4 = 28.$$

We conclude (transposing C) that

$$\boxed{A^{-1} = \frac{1}{28} \begin{bmatrix} 16 & 4 & -4 \\ -1 & 5 & 2 \\ -5 & -3 & 10 \end{bmatrix}.}$$