MATH 205 PRACTICE EXAM ON MATRIX INVERSION

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A, B be $n \times n$ invertible matrices. Which of the following equations are false?

A. $(AB)^{-1} = B^{-1}A^{-1}$.

B. $(A^\top)^{-1} = (A^{-1})^\top$.

C. $(A + B)^{-1} = A^{-1} + B^{-1}$.

D. $(A^\top B^\top)^{-1} = (A^{-1}B^{-1})^\top$.

E. None of these: the four equations hold for all invertible $n \times n$ matrices.

Answer: **C**.
Reason: The sum of two invertible matrices need not be invertible!

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an invertible matrix. Which of the following statements are false?

A. $A^\top$ and A^{-1} are both invertible.

B. If A has only integer entries, then A^{-1} has only integer entries.

C. If A is diagonal, then A^{-1} is diagonal.

D. If A is upper triangular, then A^{-1} is upper triangular.

E. None of these: the four statements are true.

Answer: **B**.
Reason: This isn't even true for 1×1 matrices.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let c be a nonzero real number and define the matrices

$$A = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of these matrices are invertible?

A. A only.

B. B only.

C. A and B only.

D. A, B, and C.

E. Other.

Answer: **D.**
Reason: Since c is nonzero, these are all elementary row operation matrices.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let a, b, c be nonzero real numbers and define the matrices

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad B = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}, \quad C = \begin{bmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix}.$$

Which of the following is equal to A^{-1}?

A. A.

B. B.

C. C.

D. D.

E. None of these.

Answer: **B.**
Reason: Multiply it out!

Problem 5. (15 pts.) Invert the matrix

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

or show that it is not invertible. Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to the auxiliary matrix:

$$\begin{bmatrix} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Apply $R_1 \to R_1 - 6R_3$ and then $R_3 \to R_3 - R_2$ to obtain

$$\begin{bmatrix} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}.$$

Apply $R_1 \to \frac{1}{2}R_1$ to finally get

$$\begin{bmatrix} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}.$$

We conclude the inverse is given by

$$\boxed{\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.}$$

Check:

$$\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 6. (15 pts.) Invert the matrix

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ 1 & 12 & 7 \end{bmatrix}$$

or show that it is not invertible. Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ to the auxiliary matrix:

$$\begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 12 & 7 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -8 & -4 & -2 & 1 & 0 \\ 0 & 8 & 4 & -1 & 0 & 1 \end{bmatrix}.$$

Apply $R_3 \to R_3 + R_2$ to obtain

$$\begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -8 & -4 & -2 & 1 & 0 \\ 0 & 8 & 4 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -8 & -4 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{bmatrix}.$$

The row of of zeros on the LHS implies the matrix is $\boxed{\text{not invertible.}}$

Indeed, from our work we can see the original matrix has linearly dependent rows since

$$R_2 + R_2 - 3R_1 = 0.$$

Problem 7. (15 pts.) Define the matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}.$$

Solve the system

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

by inverting A and left-multiplying both sides of (1) by A^{-1}.

Solution: Apply $R_3 \leftrightarrow R_1$ and then $R_2 \to R_2 - R_1$ and $R_3 - 2R_1$ to the auxiliary matrix:

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{bmatrix}.$$

Apply $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 3R_3$ and then $R_3 \to R_3 + R_2$ to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & -2 & -3 & 1 & 5 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & -2 & -3 & 1 & 5 \\ 0 & 0 & -1 & -2 & 1 & 3 \end{bmatrix}.$$

Next do $R_1 \to R_1 + 2R_3$ and $R_2 \to R_2 - 2R_3$, and then $R_3 \to -R_3$ to get

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & -2 & -3 & 1 & 5 \\ 0 & 0 & -1 & -2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & 5 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & 5 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & -3 \end{bmatrix}.$$

We can check that

$$\begin{bmatrix} -3 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12+4+15 \\ 4-2-3 \\ 8-2-9 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix}}.$$

Problem 8. (15 pts.) Define the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Use row operations to invert A, assuming that $ad - bc \neq 0$.

Solution: Simultaneously do $R_1 \to dR_1 - bR_2$ and $R_2 \to aR_2 - cR_1$ to get

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \to \begin{bmatrix} ad - bc & 0 & d & -b \\ 0 & ad - bc & -c & a \end{bmatrix}.$$

Then multiply both R_1 and R_2 by $(ad - bc)^{-1}$ to get

$$\begin{bmatrix} ad - bc & 0 & d & -b \\ 0 & ad - bc & -c & a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

We conclude

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$