MATH 205 PRACTICE EXAM ON LINEAR SYSTEMS OF EQUATIONS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

The linear system

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

has how many solutions?

A. 0.

B. 1.

C. 3.

D. Infinitely many.

E. None of these.

Answer: **D**. Reason: The equations are all multiples of the first, which has infinitely many solutions. Alternatively, consider $\mathbf{x}^T = (5, c, 2c)$ for $c \in \mathbb{R}$.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the linear system

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.$$

Which of the following solves the system?

A. $(x_1, x_2, x_3) = (2, 1, 1)$.

B. $(x_1, x_2, x_3) = (1, -1, 2)$.

C. $(x_1, x_2, x_3) = (1, 1, 2)$.

D. $(x_1, x_2, x_3) = (2, 1, -1)$.

E. None of these.

Answer: **C**.
Reason: Try the matrix multiplication.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $n \times n$ matrix and consider the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}. \tag{1}$$

Which of the following statements best describes the number of solutions of (2)?

A. It could have one, zero, or infinitely many solutions.

B. It could have one or zero solutions.

C. It could have one or infinitely many solutions.

D. It could have zero, or infinitely many solutions.

E. None of these.

Answer: **C**.
Reason: $\mathbf{x} = \mathbf{0}$ is always one solution and the only one if there are no free variables (if A is invertible). If $\exists \mathbf{v} \neq 0$ such that $A\mathbf{v} = \mathbf{0}$, then also $A(c\mathbf{v}) = \mathbf{0}$ for all $c \in \mathbb{R}$.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $m \times n$ matrix with $m < n$, let $\mathbf{b}$ be a nonzero element of $\mathbb{R}^m$, and consider the nonhomogeneous linear system

$$A\mathbf{x} = \mathbf{b}. \tag{2}$$

Which of the following statements best describes the number of solutions of (2)?

A. It could have one, zero, or infinitely many solutions.

B. It could have one or zero solutions.

C. It could have one or infinitely many solutions.

D. It could have zero, or infinitely many solutions.

E. None of these.

Answer: **D**.
Reason: Since $\mathbf{b} \neq \mathbf{0}$, zero many solutions is possible (e.g. consider $A =$ the zero matrix). Otherwise $m < n$ implies that if there is any solution, there is at least one free variable and hence infinitely many solutions.

Problem 5. (15 pts.) Use Gaussian elimination to solve the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 24 \\ 40 \end{bmatrix},$$

or to show that it has no solutions. Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 6R_1$ to the auxiliary matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 7 & 8 & 9 & 40 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & -6 & -12 & -23 \end{bmatrix}.$$

Apply $R_3 \to R_3 - 2R_2$ to obtain

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last row obtained implies the equation $0x_1+0x_2+0x_3 = 1$, which has no solutions. Therefore $\boxed{\text{no solutions exist!}}$

Problem 6. (15 pts.) Use Gaussian elimination to solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix},$$

or to show that it has no solutions. Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_2 \to R_2 - 6R_1$ and $R_3 \to R_3 - 5R_1$ to the auxiliary matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 6 & -4 & 5 & 31 \\ 5 & 2 & 2 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -10 & -1 & 19 \\ 0 & -3 & -3 & 3 \end{bmatrix}$$

Apply $R_3 \to -\frac{1}{3}R_3$ and then apply $R_2 \leftrightarrow R_3$ to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -10 & -1 & 19 \\ 0 & -3 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -10 & -1 & 19 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -10 & -1 & 19 \end{bmatrix}.$$

Apply $R_3 \to R_3 + 10R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -10 & -1 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 9 & 9 \end{bmatrix}.$$

The last row implies $9x_3 = 9$ or $x_3 = 1$. The second row asserts $x_2 + x_3 = -1$, hence $x_2 = -1 - 1 = -2$. Finally, the first row implies $x_1 + x_2 + x_3 = 2$ hence $x_1 = 2 - x_2 - x_3 = 3$. We conclude

$$\boxed{\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}.$$

Problem 7. (15 pts.) Use Gaussian elimination to solve the linear system

$$\begin{bmatrix} -1 & 1 & -1 & 3 \\ 3 & 1 & -1 & -1 \\ 2 & -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix},$$

or to show that it has no solutions. Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_1 \to -R_1$ and then $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$ to the auxiliary matrix :

$$\begin{bmatrix} -1 & 1 & -1 & 3 & -2 \\ 3 & 1 & -1 & -1 & 2 \\ 2 & -1 & -2 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 3 & 1 & -1 & -1 & 2 \\ 2 & -1 & -2 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 4 & -4 & 8 & -4 \\ 0 & 1 & -4 & 5 & 2 \end{bmatrix}.$$

Apply $R_2 \leftrightarrow R_3$ and then $R_3 \to \frac{1}{4}R_3 - R_2$:

$$\begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 4 & -4 & 8 & -4 \\ 0 & 1 & -4 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 4 & -4 & 8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 3 & -3 & -3 \end{bmatrix}.$$

Apply $R_3 \to \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 3 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -3 & 2 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

Suppose $x_4 = c \in \mathbb{R}$. The last row implies $x_3 = -1 + c$. The second row asserts $x_2 - 4x_3 + 5x_4 = 2 \implies x_2 = 4c - 4 - 5c + 2 = -2 - c$. Finally, the first row gives $x_1 - x_2 + x_3 - 3x_4 = 2 \implies x_1 = -c - 2 - c + 1 + 3c + 2 = 1 + c$. We conclude that for any $c \in \mathbb{R}$

$$\boxed{\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}},$$

is a solution.

Problem 8. (15 pts.) Use Gaussian elimination to solve the linear system

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}.$$

Show your work by labeling row operations in the notation of your choice.

Solution: Apply $R_1 \leftrightarrow R_2$ and then $R_2 \to R_2 - 2R_1$ to the auxiliary matrix :

$$\begin{bmatrix} 2 & 1 & 2 & 6 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 6 \\ 0 & 3 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 6 \\ 0 & 3 & 2 & 3 \end{bmatrix}.$$

Apply $R_3 \to R_3 + R_2$ and then $R_2 \to -\frac{1}{3}R_2$ and $R_3 \to \frac{1}{2}R_3$. :

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 6 \\ 0 & 3 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 6 \\ 0 & 0 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{9}{2} \end{bmatrix}.$$

By the latter two rows $x_3 = \frac{9}{2}$ and $x_2 = -2$. From the first row, we have $x_1 = -2x_2 - x_3 = 4 - \frac{9}{2} = -\frac{1}{2}$. Thus

$$\boxed{\mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{9}{2} \end{bmatrix}}$$

is the solution.

Problem 9. (15 pts.) Find nonzero constants a, b, c, d such that the linear system

$$\begin{bmatrix} 2 & 10 & 3 \\ a & b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ d \end{bmatrix}$$

has no solution.

Solution: If we take

$$\boxed{a = 2, \qquad b = 10, \qquad c = 3, \quad \text{and} \quad d = 1,}$$

then the equations given by the matrix identify above become

$$\begin{aligned} 2x_1 + 10x_2 + 3x_3 &= 6, \\ 2x_1 + 10x_2 + 3x_3 &= 1. \end{aligned}$$

But, evidently, this has no solutions because $6 \neq 1$.

(In general, you can take (a, b, c) to be any multiple of the first row and choose d to contradict what the first equation implies.)

Problem 10. (15 pts.) Find nonzero constants b_1, b_2, b_3 such that the linear system

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has no solution.

Solution: Observe that the rows of the leftmost matrix satisfy $2R_1 + R_2 = R_3$. In other words, adding the twice the equation $2x_1 + x_2 = b_1$ to the equation $x_1 + 3x_2 = b_2$ implies

$$5x_1 + 5x_2 = 2b_1 + b_2.$$

On the other hand the third row above implies $5x_1 + 5x_2 = b_3$. Therefore, for the system of equations to be consistent, we need $2b_1 + b_2 = b_3$. To ensure there is no solution, it suffices to choose $2b_1 + b_2 \neq b_3$. For instance (since we want none of the constants to be zero), we could take

$$\boxed{b_1 = 1, \quad b_2 = 1, \quad b_3 = 2.}$$