# MATH 205 PRACTICE EXAM ON LAPLACE TRANSFORMS

**Name:**

**Section:**

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

**Problem 1.** (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Which of the following statements best explains the Laplace transform $L[1]$ of the constant function 1?

**A**. We can think of it as $L[t^0]$ or $L[e^{0t}]$.

**B**. We can think of it as $L[t^0]$, but not as $L[e^{0t}]$.

**C**. We can think of it as $L[e^{0t}]$, but not as $L[t^0]$.

**D**. We can not think of it as $L[t^0]$ or $L[e^{0t}]$.

**E**. None of these.

Answer: **A**.
Reason: Recall the formulas

$$L[e^{at}] = \frac{1}{s-a}, \qquad L[t^n] = \frac{n!}{s^{n+1}}.$$

If we take $a = 0$ or $n = 0$ above, we find

$$L[e^{0t}] = \frac{1}{s}, \qquad L[t^0] = \frac{0!}{s} = \frac{1}{s},$$

since by convention $0! = 1$. Both recover the correct formula.

**Problem 2.** (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Which of the following functions does not have a Laplace transform?

**A**. $2t$.

**B**. $t^2$.

**C**. $e^{2t}$.

**D**. $e^{t^2}$.

**E**. All of these functions have Laplace transforms.

Answer: **D**.
Reason: The function $f(t) = e^{t^2}$ has super exponential growth. In particular, $\int_0^\infty e^{t^2 - st}\, dt$ does not have finite growth for any value of $s$.

**Problem 3.** (15 pts.) Find Laplace transforms of the following functions.

(a) $(2+e^{5t})t^3$.

**Solution:**

$$\begin{aligned} L[(2+e^{5t})t^3](s) &= 2L[t^3](s) + L[e^{5t}t^3](s) \\ &= 2\cdot\frac{6}{s^4} + L[t^3](s-5) \\ &= \frac{12}{s^4} + \frac{6}{(s-5)^4}. \end{aligned}$$

(b) $e^{\alpha t}(k\cos(\beta t) + m\sin(\beta t))$, where $\alpha$, $\beta$, $k$, $m$ are real numbers.

**Solution:**

$$\begin{aligned} L[e^{\alpha t}\big(k\cos(\beta t) + m\sin(\beta t)\big)](s) &= kL[\cos(\beta t)](s-\alpha) + mL[\sin(\beta t)](s-\alpha) \\ &= k\frac{s-\alpha}{(s-\alpha)^2+\beta^2} + m\frac{\beta}{(s-\alpha)^2+\beta^2}. \end{aligned}$$

(c) $f(t) = \begin{cases} 0 & \text{when } 0 \le t < 1, \\ t^2 - 2t + 2 & \text{when } t \ge 1. \end{cases}$

**Solution:** Write

$$f(t) = u_1(t)(t^2-2t+2).$$

Then

$$\begin{aligned} L[u_1(t)(t^2-2t+2)](s) &= e^{-s}L[(t+1)^2 - 2(t+1) + 2](s) \\ &= e^{-s}L[t^2+1](s) \\ &= e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right). \end{aligned}$$

**Problem 4.** (15 pts.) Find inverse Laplace transforms of the following functions.

(a) $\dfrac{3}{s^2+4}$.

**Solution:**

$$L^{-1}\left[\frac{3}{s^2+4}\right] = \frac{3}{2}\sin(2t).$$

(b) $\dfrac{2s-3}{s^2-4}$

**Solution:** Partial fraction decomposition

$$\frac{2s-3}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} \implies 2s-3 = (A+B)s + 2A - 2B$$

We solve to $A, B$ to get

$$\frac{2s-3}{s^2-4} = \frac{1}{4}\frac{1}{s-2} + \frac{7}{4}\frac{1}{s+2}.$$

Then

$$\begin{aligned} L^{-1}[\frac{2s-3}{s^2-4}] &= \frac{1}{4}L^{-1}\left[\frac{1}{s-2}\right] + \frac{7}{4}L^{-1}\left[\frac{1}{s+2}\right] \\ &= \frac{1}{4}e^{2t} + \frac{7}{4}e^{-2t}. \end{aligned}$$

(c) $\dfrac{1-2s}{s^2+4s+5}$

**Solution:** Complete the square:

$$\frac{1-2s}{s^2+4s+5} = \frac{1-2s}{(s+2)^2+1} = \frac{5}{(s+2)^2+1} - \frac{2(s+2)}{(s+2)^2+1}.$$

Then

$$\begin{aligned} L^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] &= 5L^{-1}\left[\frac{1}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] \\ &= 5e^{-2t}\sin(t) - 2e^{-2t}\cos(t). \end{aligned}$$

**Problem 5.** (15 pts.) Use the method of Laplace transforms to solve the initial value problem

$$y'' - 9y' + 20y = 0, \qquad y(0) = 3, \quad y'(0) = 10.$$

**Solution:** Laplace transform both sides:

$$L[y'' - 9y' + 20y] = 0.$$

Expand the left hand side:

$$s^2 L[y] - sy(0) - y'(0) - 9\big(sL[y] - y(0)\big) + 20L[y] = 0.$$

Collect terms and Solve for $L[y]$:

$$(s^2 - 9s + 20)L[y] = 3s - 17,$$

$$\implies L[y] = \frac{3s - 17}{(s-4)(s-5)}$$

Using partial fraction decomposition:

$$\overset{\text{PFD}}{\implies} L[y] = \frac{5}{s-4} - \frac{2}{s-5}$$

Now inverse transform:

$$y = L^{-1}\left[\frac{5}{s-4} - \frac{2}{s-5}\right] = 5e^{4t} - 2e^{5t}.$$

**Problem 6.** Use the method of Laplace transforms to solve the initial value problem

$$y'' - y = 2\sin(t), \qquad y(0) = -1, \quad y'(0) = 0.$$

**Solution:** Laplace transform both sides:

$$L[y'' - y] = L[2\sin t] = \frac{2}{s^2+1}.$$

Expand the left hand side:

$$s^2 L[y] - sy(0) - y'(0) - L[y] = \frac{2}{s^2+1}.$$

Collect terms and Solve for $L[y]$:

$$\begin{aligned}(s^2-1)L[y] &= \frac{2}{s^2+1} - s,\\ \implies L[y] &= \frac{2}{(s^2+1)(s-1)(s+1)} - \frac{s}{(s-1)(s+1)}.\end{aligned}$$

Using partial fraction decomposition:

$$\overset{\text{PFD}}{\Longrightarrow} L[y] = -\frac{1}{s+1} - \frac{1}{s^2+1}.$$

Now inverse transform:

$$y = L^{-1}\left[-\frac{1}{s+1} - \frac{1}{s^2+1}\right] = -e^{-t} - \sin(t).$$

**Problem 7.** (15 pts.) Use the method of Laplace transforms to solve the initial value problem

$$y'' + 2y' + 5y = 6e^{-t}\cos(t), \qquad y(0) = 1, \quad y'(0) = 1.$$

Hint: use the fact that

$$\frac{6(s+1)}{((s+1)^2+1)((s+1)^2+4)} = \frac{2(s+1)}{(s+1)^2+1} - \frac{2(s+1)}{(s+1)^2+4}.$$

**Solution:** Laplace transform both sides:

$$L[y'' + 2y' + 5y] = L[6e^{-t}\cos t] = \frac{6(s+1)}{(s+1)^2+1}.$$

Expand the left hand side:

$$s^2L[y] - sy(0) - y'(0) + 2(sL[y] - y(0)) - 5L[y] = \frac{6(s+1)}{(s+1)^2+1}.$$

Collect terms and Solve for $L[y]$:

$$((s+1)^2+4)L[y] = \frac{6(s+1)}{(s+1)^2+1} + s + 3,$$

$$\implies L[y] = \frac{6(s+1)}{((s+1)^2+1)((s+1)^2+4)} + \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}.$$

Using partial fraction decomposition:

$$\overset{\text{PFD}}{\implies} L[y] = \frac{2(s+1)}{(s+1)^2+1} - \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}.$$

Now inverse transform:

$$\begin{aligned} y &= L^{-1}\left[\frac{2(s+1)}{(s+1)^2+1} - \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}\right] \\ &= 2e^{-t}\cos(t) - e^{-t}\cos(2t) + e^{-t}\sin(2t). \end{aligned}$$

**Problem 8.** (15 pts.) Solve the initial value problem

$$y'' + 4y = \begin{cases} t & \text{for } 0 \le t < \pi/2, \\ \pi/2 & \text{for } \pi/2 \le t < \infty, \end{cases} \qquad y(0) = y'(0) = 0.$$

**Solution:** Write the piecewise function as

$$f(t) = t(1 - u_{\frac{\pi}{2}}(t)) + \frac{\pi}{2}u_{\frac{\pi}{2}}(t) = t - (t - \frac{\pi}{2})u_{\frac{\pi}{2}}(t).$$

Laplace transform both sides:

$$L[y'' + 4y] = L[f(t)] = \frac{1}{s^2}(1 - e^{-\frac{\pi}{2}s}).$$

Expand the left hand side:

$$s^2L[y] - sy(0) - y'(0) + 4L[y] = \frac{1}{s^2}(1 - e^{-\frac{\pi}{2}s}).$$

Collect terms and Solve for $L[y]$:

$$(s^2 + 4)L[y] = \frac{1}{s^2}(1 - e^{-\frac{\pi}{2}s}),$$

$$\implies L[y] = \frac{1}{s^2(s^2 + 4)}(1 - e^{-\frac{\pi}{2}s}).$$

Using partial fraction decomposition:

$$\overset{\text{PFD}}{\implies} L[y] = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2 + 4} - \frac{1}{s^2}e^{-\frac{\pi}{2}s} + \frac{1}{s^2 + 4}e^{-\frac{\pi}{2}s}\right).$$

Now inverse transform:

$$\begin{aligned} y &= \frac{1}{4}L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2 + 4} - \frac{1}{s^2}e^{-\frac{\pi}{2}s} + \frac{1}{s^2 + 4}e^{-\frac{\pi}{2}s}\right] \\ &= \frac{1}{4}(t - \frac{1}{2}\sin 2t) - \frac{1}{4}\Big(t - \frac{\pi}{2} - \frac{1}{2}\sin(2t - \pi)\Big)u_{\frac{\pi}{2}}(t) \\ &= \frac{1}{4}(t - \frac{1}{2}\sin 2t) - \frac{1}{4}\Big(t - \frac{\pi}{2} + \frac{1}{2}\sin(2t)\Big)u_{\frac{\pi}{2}}(t). \end{aligned}$$

**Problem 9.** (15 pts.) Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi), \qquad y(0) = y'(0) = 0,$$

where $\delta$ is the Dirac impulse function.

**Solution:** Laplace transform both sides:

$$L[y'' + 2y' + 2y] = L[\delta(t - \pi)] = e^{-\pi s}.$$

Expand the left hand side:

$$s^2 L[y] - sy(0) - y'(0) + 2(sL[y] - y(0)) + 2L[y] = e^{-\pi s}.$$

Collect terms and Solve for $L[y]$:

$$(s^2 + 2s + 2)L[y] = e^{-\pi s},$$

$$\implies L[y] = \frac{1}{s^2 + 2s + 2} e^{-\pi s}.$$

Completing the square:

$$\overset{\text{CTS}}{\implies} L[y] = \frac{1}{(s+1)^2 + 1} e^{-\pi s}.$$

Now inverse transform:

$$\begin{aligned} y &= L^{-1}\left[\frac{1}{(s+1)^2 + 1} e^{-\pi s}\right] \\ &= e^{-t+\pi} \sin(t - \pi) u_\pi(t) \\ &= -e^{\pi - t} \sin(t) u_\pi(t). \end{aligned}$$