MATH 205 PRACTICE EXAM ON LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Fix an $n \times n$ matrix A with real entries and a vector $\mathbf{b}$ of functions of t. Consider the differential equations

$$\text{(H)} \qquad \mathbf{x} = A\mathbf{x}',$$

$$\text{(NH)} \qquad \mathbf{x} = A\mathbf{x}' + \mathbf{b}.$$

Which of the following is true about the solutions of (H) and (NH)?

A. The solutions of (H) form a vector space; the solutions of (NH) form another vector space.

B. The solutions of (H) form a vector space; the solutions of (NH) do not.

C. The solutions of (H) do not form a vector space; the solutions of (NH) do.

D. The solutions of (H) do not form a vector space; neither do the solutions of (NH).

E. None of these.

Answer: **B**.
Reason: Solutions of the homogeneous problem are indeed closed under scalar multiplication and addition, but you showed in the homework that solutions of the non-homogeneous problem are not.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $n \times n$ matrix, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbf{R}^n$, and let c, d be nonzero real numbers. If $\mathbf{u} + i\mathbf{v}$ is an eigenvector of A with eigenvalue $c + di$, then what can we say about $\mathbf{u} - i\mathbf{v}$?

A. It is an eigenvector of A with eigenvalue $c + di$.

B. It is an eigenvector of A with eigenvalue $d + ci$.

C. It is an eigenvector of A with eigenvalue $c - di$.

D. It is an eigenvector of A with eigenvalue $d - ci$.

E. None of these.

Answer: **C.**
Reason: Taking the complex conjugation of $A(\mathbf{u} + i\mathbf{v}) = (c + di)(\mathbf{u} + i\mathbf{v})$ shows that $A(\mathbf{u} - i\mathbf{v}) = (c - di)(\mathbf{u} - i\mathbf{v})$ (since A has real entries).

Problem 3. (15 pts.) Define the matrix

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}.$$

(a) Solve (Find all solutions of) the differential equation $\mathbf{x}' = A\mathbf{x}$.

Solution: We first compute that $p(\lambda) = (-2-\lambda)(5-\lambda) + 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$. It follows that $\lambda_1 = 4$ and $\lambda_2 = -1$ are the eigenvalues.

For the eigenvectors, we compute

$$A - 4I = \begin{bmatrix} -6 & 3 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ 4-eigenvector.}$$

And

$$A + I = \begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ (}-1\text{)-eigenvector.}$$

Therefore the general solution of the first order system is

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(b) Find the particular solution of $\mathbf{x}' = A\mathbf{x}$ which satisfies

$$\mathbf{x}(0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Solution: Using the general solution above, we compute

$$\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Therefore $c_2 = -2c_1$ and $c_1 + 3c_2 = -2$. We conclude the solution is given as above with $c_1 = 2/5$ and $c_2 = -4/5$.

Problem 4. (15 pts.) Define the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

(a) Solve (Find all solutions of) the differential equation $\mathbf{x}' = A\mathbf{x}$.

Solution: We first compute that $p(\lambda) = (-1-\lambda)(-1-\lambda) + 4 = \lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i)$. It follows that $\lambda_1 = -1 - 2i$ and $\lambda_2 = -1 + 2i$ are the eigenvalues.

For the eigenvectors, we compute

$$A - (-1+2i)I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (-1+2i)\text{-eigenvector.}$$

Since the eigenvalues are complex conjugates, the other eigenvector must be the conjugate:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

One way to correctly express the general solution is as the complex-valued solution

$$\mathbf{x}(t) = \tilde{c}_1 e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + \tilde{c}_2 e^{(-1-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

where the constants $\tilde{c}_1, \tilde{c}_2$ are in general complex numbers.

However, we are after the real-valued solution. Considering the real and imaginary parts, $\lambda_1 = a + bi = -1 + 2i$ and $\mathbf{v}_1 = \mathbf{r} + i\mathbf{s} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the real version of the geneal solution is

$$\mathbf{x}(t) = c_1 e^{-t} \left(\cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^{-t} \left(\sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

where c_1, c_2 are real constants. (which can be found by taking the real and imaginary parts of the complex solution).

(b) Find the particular solution of $\mathbf{x}' = A\mathbf{x}$ which satisfies

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution: Using the general solution above, we compute

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, the solution is given as above with $c_1 = 3$ and $c_2 = 2$.

Problem 5. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 8 & -5 \\ 10 & 6 \end{bmatrix} \mathbf{u} + 4t^3 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + 8t^7 e^{7t} \begin{bmatrix} 7\cos(7t) + \sin(7t) \\ 10\sin(7t) \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $\lambda^2 - 14\lambda + 98$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = t^4 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + t^8 e^{7t} \begin{bmatrix} 7\cos(7t) + \sin(7t) \\ 10\sin(7t) \end{bmatrix}.$$

Solution: Because we have a particular solution given, we need only find the general solution to the homogeneous system $\mathbf{u}' = A\mathbf{u} = \begin{bmatrix} 8 & -5 \\ 10 & 6 \end{bmatrix} \mathbf{u}$ to find the general solution to the given nonhomogeneous system.

$$p(\lambda) = \lambda^2 - 14\lambda + 98 = 0$$
$$\implies \lambda = 7 \pm 7i.$$

$$A - (7+7i)I = \begin{bmatrix} 1-7i & -5 \\ 10 & -1-7i \end{bmatrix} \sim \begin{bmatrix} 10 & -1-7i \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix} + i \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

Similarly, for $\lambda = 7 - 7i$, we have

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 10 \end{bmatrix} - i \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

Thus, the general solution to the homogeneous system $\mathbf{u}' = A\mathbf{u}$ is

$$\mathbf{u}_h = c_1 e^{7t} \left(\cos(7t) \begin{bmatrix} 1 \\ 10 \end{bmatrix} - \sin(7t) \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right) + c_2 e^{7t} \left(\sin(7t) \begin{bmatrix} 1 \\ 10 \end{bmatrix} + \cos(7t) \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right).$$

Finally, we have the general solution to the given nonhomogeneous system

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_h + \mathbf{u}_p \\ &= c_1 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} \sin(7t) + 7\cos(7t) \\ 10\sin(7t) \end{bmatrix} \\ &\quad + t^4 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + t^8 e^{7t} \begin{bmatrix} 7\cos(7t) + \sin(7t) \\ 10\sin(7t) \end{bmatrix} \end{aligned}$$

Problem 6. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 15 & 8 & -4 \\ 4 & 11 & 4 \\ 4 & 8 & 7 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 5 \\ 14 \\ 5 \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda-3)(\lambda-11)(\lambda-19)$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Solution: Because we have a particular solution given, we need only find the general solution to the homogeneous system $\mathbf{u}' = A\mathbf{u} = \begin{bmatrix} 15 & 8 & -4 \\ 4 & 11 & 4 \\ 4 & 8 & 7 \end{bmatrix} \mathbf{u}$.

$$\begin{gathered} p(\lambda) = -(\lambda-3)(\lambda-11)(\lambda-19) = 0 \\ \implies \lambda = 3, \lambda = 11, \lambda = 19. \end{gathered}$$

For $\lambda = 3$, we have

$$A - 3I = \begin{bmatrix} 12 & 8 & -4 \\ 4 & 8 & 4 \\ 4 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda = 11$, we have

$$A - 11I = \begin{bmatrix} 4 & 8 & -4 \\ 4 & 0 & 4 \\ 4 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 19$, we have

$$A - 19I = \begin{bmatrix} -4 & 8 & -4 \\ 4 & -8 & 4 \\ 4 & 8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Putting this together, the general solution to the homogeneous system $\mathbf{u}' = A\mathbf{u}$ is

$$\mathbf{u}_h = c_1 \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{11t} \\ e^{11t} \\ e^{11t} \end{bmatrix} + c_3 \begin{bmatrix} e^{19t} \\ e^{19t} \\ e^{19t} \end{bmatrix}.$$

Finally, we have the general solution to the nonhomogeneous system

$$\mathbf{u} = \mathbf{u}_h + \mathbf{u}_p = \begin{bmatrix} c_1 e^{3t} - c_2 e^{11t} + c_3 e^{19t} + 1 \\ -c_1 e^{3t} + c_2 e^{11t} + c_3 e^{19t} - 2 \\ c_1 e^{3t} + c_2 e^{11t} + c_3 e^{19t} + 1 \end{bmatrix}$$

Problem 7. Let α, β be nonzero real constants. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{bmatrix} \mathbf{u}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda - (\alpha+\beta))^2(\lambda - (\alpha - 2\beta))$.

Solution: Let $A = \begin{bmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{bmatrix}$

$$\chi(\lambda) = -(\lambda - (\alpha+\beta))^2(\lambda - (\alpha - 2\beta)) = 0$$
$$\implies \lambda = \alpha + \beta \text{ (mult. 2)}, \ \lambda = \alpha - 2\beta.$$

For $\lambda = \alpha + \beta$, we have

$$A - (\alpha+\beta)I = \begin{bmatrix} -\beta & -\beta & -\beta \\ -\beta & -\beta & -\beta \\ -\beta & -\beta & -\beta \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = \alpha - 2\beta$, we have

$$A - (\alpha - 2\beta)I = \begin{bmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the general solution is

$$\mathbf{u} = \begin{bmatrix} -c_1 e^{(\alpha+\beta)t} - c_2 e^{(\alpha+\beta)t} + c_3 e^{(\alpha-2\beta)t} \\ c_1 e^{(\alpha+\beta)t} + c_3 e^{(\alpha-2\beta)t} \\ c_2 e^{(\alpha+\beta)t} + c_3 e^{(\alpha-2\beta)t} \end{bmatrix}.$$

Problem 8. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \mathbf{u} + 6t^5e^t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 7t^6e^t \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 8t^7e^{7t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda - 1)^2(\lambda - 7)$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = t^6e^t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t^7e^t \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t^8e^{7t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution: Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$. Because we are given a particular solution to the nonhomogeneous system, we need only find the general solution for the homogeneous system $\mathbf{u}' = A\mathbf{u}$.

$$\begin{aligned} \chi(\lambda) &= -(\lambda - 1)^2(\lambda - 7) = 0 \\ &\implies \lambda = 1 \text{ (mult. 2)},\ \lambda = 7. \end{aligned}$$

For $\lambda = 1$, we have

$$A - I = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 7$, we have

$$A - 7I = \begin{bmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Putting this together, the general solution to $\mathbf{u}' = A\mathbf{u}$ is

$$\mathbf{u}_h = c_1e^t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3e^{7t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Finally, we have the general solution to the nonhomogeneous system

$$\mathbf{u} = \mathbf{u}_h + \mathbf{u}_p = \begin{bmatrix} -2c_1e^t - c_2e^t + c_3e^{7t} + 2t^6e^t + t^8e^{7t} \\ c_1e^t + 2c_3e^{7t} - t^6e^t - t^7e^t + 2t^8e^{7t} \\ c_2e^t + 2t^7e^t + t^8e^{7t} \end{bmatrix}.$$

Problem 9. Consider the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}. \tag{1}$$

(a) Find a fundamental matrix of solutions for the homogeneous linear system corresponding to (1).

Solution:

Step 1: We compute $p(\lambda) = (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$. Hence $\lambda_1 = -3$ and $\lambda_2 = -1$.

Step 2: We find the eigenvectors:

$$A + 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (-3)\text{-eigenvector},$$

$$A + I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (-1)\text{-eigenvector}.$$

Step 3: We conclude that we have a fundamental basis of solutions given by

$$\mathbf{x}_1(t) = e^{\lambda_1}\mathbf{v}_1 = \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}, \qquad \mathbf{x}_2(t) = e^{\lambda_2}\mathbf{v}_2 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

The fundamental matrix is given by

$$\mathbf{X} = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}.$$

(continued →)

(b) Use the method of variation of parameters to find one particular solution of (1).

Solution:

Step 1: We begin by solving

$$\mathbf{X}\mathbf{u}' = \mathbf{b} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

for $\mathbf{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$. Since $\det(\mathbf{X}) = 2e^{-2t}$, we observe

$$\mathbf{X}^{-1} = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}.$$

Hence

$$\mathbf{u}' = \mathbf{X}^{-1}\mathbf{b} = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{bmatrix}.$$

Step 2: We compute that

$$u_1 = \int e^{2t} - \frac{3}{2}te^{3t}\,dt = \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t} - \frac{1}{2}te^{3t},$$
$$u_2 = \int 1 + \frac{3}{2}te^{t}\,dt = t - \frac{3}{2}e^{t} + \frac{3}{2}e^{t}.$$

In general, by integration by parts: $\int te^{at}\,dt = -\frac{1}{a^2}e^{at} + \frac{1}{a}te^{at}$.

Step 3: Finally, we compute

$$\mathbf{x}_p = \mathbf{X}\mathbf{u} = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t} - \frac{1}{2}te^{3t} \\ t - \frac{3}{2}e^{t} + \frac{3}{2}e^{t} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} + t + \frac{1}{2}e^{-t} + te^{-t} \\ -\frac{5}{3} + 2t - \frac{1}{2}e^{-t} + te^{-t} \end{bmatrix}$$

(c) Find the general solution (all solutions) of the linear system (1).

Solution: The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_p$ are given as above.

Problem 10. Consider the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}. \tag{2}$$

(a) Find a fundamental matrix of solutions for the homogeneous linear system corresponding to (2).

Solution:

Step 1: Let $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$. We compute the characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = (2-\lambda)(-2-\lambda) + 5 = \lambda^2 - 4 + 5 = \lambda^2 + 1.$$

So the eigenvalues are complex: $\lambda = \pm i$.

Step 2: Find an eigenvector for $\lambda = i$:

$$A - iI = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \sim \begin{bmatrix} 1 & -2-i \\ 0 & 0 \end{bmatrix} \implies \mathbf{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}.$$

The eigenvector for $\lambda = -i$ is the conjugate.

Step 3: Using that $\lambda = a + bi = i$ and $\mathbf{v} = \mathbf{r} + i\mathbf{s} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we fundamental solutions find:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix},$$

$$\mathbf{x}_2(t) = \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}.$$

Hence, a fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix}.$$

Alternatively

$$\mathbf{X}(t) = \begin{bmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & 2\sin t - \cos t \end{bmatrix}.$$

(continued →)

(b) Use the method of variation of parameters to find one particular solution of (2).

Solution:

Step 1: We begin by solving

$$\mathbf{X}\mathbf{u}' = \mathbf{b} = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

for $\mathbf{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$. Since $\det(\mathbf{X}) = -1$, we observe

$$\mathbf{X}^{-1} = \begin{bmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & -2\cos t + \sin t \end{bmatrix}.$$

Hence

$$\mathbf{u}' = \mathbf{X}^{-1}\mathbf{b} = \begin{bmatrix} -\sin t & 2\sin t + \cos t \\ \cos t & -2\cos t + \sin t \end{bmatrix} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} 2\cos t \sin t + 2\sin^2 t \\ \sin^2 t - \cos^2 t - 2\cos t \sin t \end{bmatrix}$$
$$= \begin{bmatrix} \sin 2t + 1 - \cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix}$$

Step 2: We compute that

$$u_1 = \int \sin 2t + 1 - \cos 2t \, dt = t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t,$$
$$u_2 = \int -\cos 2t - \sin 2t \, dt = -\frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t.$$

Step 3: Finally, we compute

$$\mathbf{x}_p = \mathbf{X}\mathbf{u} = \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t \\ -\frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t \end{bmatrix}.$$

There are a few possible equivalent ways of presenting this answer correctly!

(c) Find the general solution (all solutions) of the linear system (2).

Solution: The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the real and imaginary parts of the complex exponential solution as described in (a), and $\mathbf{x}_p(t)$ is the particular solution computed using variation of parameters in (b).

Problem 11. Convert the second-order linear differential equation

$$y'' + 5t^2 y' - y = \sin(2t)$$

into a first-order linear system.

Solution: Let $x_1 = y$ and $x_2 = x_1' = y'$. Then the following first-order linear system is equivalent to the given DE:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5t^2 x_2 + x_1 + \sin(2t) \end{aligned}$$