MATH 205 PRACTICE EXAM ON LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Fix an $n \times n$ matrix A with real entries and a vector $\mathbf{b}$ of functions of t. Consider the differential equations

$$\text{(H)} \qquad \mathbf{x} = A\mathbf{x}',$$
$$\text{(NH)} \qquad \mathbf{x} = A\mathbf{x}' + \mathbf{b}.$$

Which of the following is true about the solutions of (H) and (NH)?

A. The solutions of (H) form a vector space; the solutions of (NH) form another vector space.

B. The solutions of (H) form a vector space; the solutions of (NH) do not.

C. The solutions of (H) do not form a vector space; the solutions of (NH) do.

D. The solutions of (H) do not form a vector space; neither do the solutions of (NH).

E. None of these.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A be an $n \times n$ matrix, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbf{R}^n$, and let c, d be nonzero real numbers. If $\mathbf{u} + i\mathbf{v}$ is an eigenvector of A with eigenvalue $c + di$, then what can we say about $\mathbf{u} - i\mathbf{v}$?

A. It is an eigenvector of A with eigenvalue $c + di$.

B. It is an eigenvector of A with eigenvalue $d + ci$.

C. It is an eigenvector of A with eigenvalue $c - di$.

D. It is an eigenvector of A with eigenvalue $d - ci$.

E. None of these.

Problem 3. (15 pts.) Define the matrix

$$A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}.$$

(a) Solve (Find all solutions of) the differential equation $\mathbf{x}' = A\mathbf{x}$.

(b) Find the particular solution of $\mathbf{x}' = A\mathbf{x}$ which satisfies

$$\mathbf{x}(0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Problem 4. (15 pts.) Define the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

(a) Solve (Find all solutions of) the differential equation $\mathbf{x}' = A\mathbf{x}$.

(b) Find the particular solution of $\mathbf{x}' = A\mathbf{x}$ which satisfies

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Problem 5. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 8 & -5 \\ 10 & 6 \end{bmatrix} \mathbf{u} + 4t^3 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + 8t^7 e^{7t} \begin{bmatrix} 7\cos(7t) + \sin(7t) \\ 10\sin(7t) \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $\lambda^2 - 14\lambda + 98$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = t^4 e^{7t} \begin{bmatrix} \cos(7t) - 7\sin(7t) \\ 10\cos(7t) \end{bmatrix} + t^8 e^{7t} \begin{bmatrix} 7\cos(7t) + \sin(7t) \\ 10\sin(7t) \end{bmatrix}.$$

Problem 6. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 15 & 8 & -4 \\ 4 & 11 & 4 \\ 4 & 8 & 7 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 5 \\ 14 \\ 5 \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda - 3)(\lambda - 11)(\lambda - 19)$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 7. Let α, β be nonzero real constants. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{bmatrix} \mathbf{u}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda - (\alpha + \beta))^2(\lambda - (\alpha - 2\beta))$.

Problem 8. Find the general solution of the system

$$\mathbf{u}' = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \mathbf{u} + 6t^5 e^t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 7t^6 e^t \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 8t^7 e^{7t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

of differential equations, using the fact that the characteristic polynomial of the coefficient matrix is $-(\lambda - 1)^2(\lambda - 7)$, and the fact that one particular solution is

$$\mathbf{u}_p(t) = t^6 e^t \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t^7 e^t \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + t^8 e^{7t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Problem 9. Consider the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}. \tag{1}$$

(a) Find a fundamental matrix of solutions for the homogeneous linear system corresponding to (1).

(continued →)

(b) Use the method of variation of parameters to find one particular solution of (1).

(c) Find the general solution (all solutions) of the linear system (1).

Problem 10. Consider the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}. \tag{2}$$

(a) Find a fundamental matrix of solutions for the homogeneous linear system corresponding to (2).

(continued →)

(b) Use the method of variation of parameters to find one particular solution of (2).

(c) Find the general solution (all solutions) of the linear system (2).

Problem 11. Convert the second-order linear differential equation

$$y'' + 5t^2y' - y = \sin(2t)$$

into a first-order linear system.