MATH 205 PRACTICE EXAM ON OSCILLATIONS OF A MECHANICAL SYSTEM

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$my'' + cy' + ky = 0$$

for a spring-mass system with mass $m > 0$, damping constant $c > 0$, and spring constant $k > 0$. Which of the following statements about the system is false?

A. Motion is always oscillating, regardless of the values of m, c, k.

B. The motion is underdamped if $c^2 < 4km$.

C. The motion is critically damped if $c^2 = 4km$.

D. The motion is overdamped if $c^2 > 4km$.

E. None of these: all four statements are true.

Answer: **A**.
Reason: If the system if critically damped over overdamped, it does not exhibit oscillatory behavior.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$my'' + ky = A\cos(\alpha t) + B\sin(\alpha t)$$

for an undamped spring-mass system with mass $m > 0$, spring constant $k > 0$, and forcing function $A\cos(\alpha t) + B\sin(\alpha t)$, where A, B, α are real constants. If resonance occurs in the system, then which of the following is the general solution of the differential equation?

A. $y(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}Bt\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}At\sin\left(\sqrt{\frac{k}{m}}t\right)$.

B. $y(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{1}{2m}\sqrt{\frac{m}{k}}Bt\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}At\sin\left(\sqrt{\frac{k}{m}}t\right)$.

C. $y(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}At\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}Bt\sin\left(\sqrt{\frac{k}{m}}t\right)$.

D. $y(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{1}{2m}\sqrt{\frac{m}{k}}At\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{1}{2m}\sqrt{\frac{m}{k}}Bt\sin\left(\sqrt{\frac{k}{m}}t\right)$.

E. None of these.

Answer: **B**.
Reason: Let $P(D) = mD^2 + k$ and $\omega_0 = \sqrt{\frac{k}{m}}$. Resonance occurs when $\alpha = \omega_0$. Observe/recall that

$$P(D)\Big(\cos(\omega_0 t)\Big) = P(D)\Big(\sin(\omega_0 t)\Big) = 0.$$

The homogeneous term of each multiple choice answer is correct. On the other hand

$$\begin{aligned} P(D)\left(t\cos(\omega_0 t)\right) &= -2m\omega_0\sin(\omega_0 t) + tP(D)\Big(\sin(\omega_0 t)\Big) \\ &= -2m\omega_0\sin(\omega_0 t), \\ P(D)\left(t\sin(\omega_0 t)\right) &= 2m\omega_0\cos(\omega_0 t) + tP(D)\Big(\cos(\omega_0 t)\Big) \\ &= 2m\omega_0\cos(\omega_0 t). \end{aligned}$$

It follows that

$$P(D)\left(-\frac{1}{2m\omega_0}Bt\cos(\omega_0 t) + \frac{1}{2m\omega_0}At\sin(\omega_0 t)\right) = B\sin(\omega_0 t) + A\cos(\omega_0 t).$$

Thus the answer is B.

Problem 3. (20 pts.) A spring-mass system has an object of mass 2 kg, which hangs from a spring with spring constant $5\,\mathrm{kg/s^2}$, and which is subject to damping with constant $6\,\mathrm{kg/s}$. Let $y(t)$ be the position in meters below equilibrium position of the object at time t seconds. At time 0 the object is stretched 2 m below equilibrium position and released with an initial velocity of 1 m/s upward.

(a) Write an initial value problem consisting of a differential equation which relates $y(t)$ and its derivatives, and initial conditions which describe the initial position and initial velocity of the object.

Solution: The function $y(t)$ satisfies
$$2y'' + 6y' + 5y' = 0$$
with
$$y(0) = 2, \qquad y'(0) = -1.$$
where the signs are chosen to reflect that $y(t)$ is positive in the *below equilibrium* direction.

(b) Solve the above initial value problem to give an explicit formula for the position of the object at time t.

Solution: Setting $P(D) = D^2 + 3D + \frac{5}{2}$, the ODE is $P(D)y = 0$. The roots of $P(r)$ are given by
$$r = \frac{-3 \pm \sqrt{3^2 - 4\frac{5}{2}}}{2} = -\frac{3}{2} \pm \frac{i}{2}.$$
This means the general solution of $P(D)y = 0$ is given by
$$y(t) = e^{-\frac{3}{2}t}\left(C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2}\right).$$
Since
$$y'(t) = e^{-\frac{3}{2}t}\left(\left(\frac{1}{2}C_2 - \frac{3}{2}C_1\right)\cos\frac{t}{2} - \left(\frac{1}{2}C_1 + \frac{3}{2}C_2\right)\sin\frac{t}{2}\right).$$
The initial value conditions are
$$2 = y(0) = C_1$$
and
$$-1 = y'(0) = \frac{1}{2}C_2 - \frac{3}{2}C_1,$$
which implies $C_2 = 4$.

We conclude the solution of the IVP is
$$y(t) = e^{-\frac{3}{2}t}\left(2\cos\frac{t}{2} + 4\sin\frac{t}{2}\right).$$

(continued $\rightarrow$)

(c) Describe the long-term behavior of the object as t approaches infinity.

Solution: The system is overdamped so the solution must satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

We can also see this in the form of the solution found in (b).

Problem 4. (20 pts.) Consider a spring-mass system whose motion is governed by the differential equation

$$y'' + 16y = 60\cos(2t) + 60\sin(2t). \tag{1}$$

(a) Find the complementary solution of (1), that is, the general solution of the homogeneous differential equation corresponding to (1).

Solution: Let $P(D) = D^2 + 16$. The homogeneous problem is $P(D)y = 0$. Since $P(r)$ has roots

$$r = \pm 4i$$

the solution of the homogenous equation is

$$y_c(t) = C_1\cos(4t) + C_2\sin(4t).$$

(b) Use the method of undetermined coefficients to find a particular solution of (1).

Solution: The annihilator polynomial of the right hand side is $A(r) = r^2 + 4$, which does not share any roots with $P(r)$ above. We look for a trial solution

$$y_p(t) = A_0\cos(2t) + B_0\sin(2t).$$

We find

$$y_p''(t) = -4A_0\cos(2t) - 4B_0\sin(2t)$$

which means

$$y_p''(t) + 16y_p(t) = 12A_0\cos(2t) + 12B_0\sin(2t).$$

We conclude $A_0 = B_0 = 5$ and so

$$y_p(t) = 5\cos(2t) + 5\sin(2t).$$

(continued →)

(c) Find the general solution of (1).

Solution: The general solution is the sum of the particular solution and the complementary solution. Here it is given by

$$y(t) = C_1 \cos(4t) + C_2 \sin(4t) + 5\cos(2t) + 5\sin(2t).$$

(d) The object in the spring-mass system has a period of motion which does not depend upon any initial conditions. Find this period of motion.

Solution: The two periods of motion related to the system are the natural one $T_n = \frac{2\pi}{\omega_0} = \frac{2\pi}{4} = \frac{\pi}{2}$ and the driven one $T_d = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$. Since $T_d = 2T_n$, we see the motion is oscillatory with a period of $T = \pi$, independent of the initial conditions (which determine C_1, C_2).

Problem 5. (20 pts.) Consider a spring-mass system whose motion is governed by the differential equation and initial conditions

$$y'' + 5y' + 6y = 0, \qquad y(0) = -1, \quad y'(0) = 4. \tag{2}$$

(a) Solve the initial value problem (2).

Solution: Setting $P(D) = D^2 + 5D + 6$, the ODE is $P(D)y = 0$. The roots of $P(r)$ are given by

$$r = -2, -3$$

This means the general solution of $P(D)y = 0$ is given by

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}.$$

Since

$$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t}$$

The initial value conditions are

$$-1 = y(0) = C_1 + C_2$$

and

$$4 = y'(0) = -2C_1 - 3C_2.$$

Solving this system gives $C_1 = 1$, $C_2 = -2$.

We conclude the solution of the IVP is

$$y(t) = e^{-2t} - 2e^{-3t}.$$

(continued →)

(b) Is the system (2) underdamped, critically damped, or overdamped?

Solution: It is overdamped (two distinct real roots).

(c) Find the time at which the mass passes through the equilibrium position.

Solution: We solve

$$0 = y(t) = e^{-2t} - 2e^{-3t},$$

for $t > 0$. We find

$$2 = e^t$$

which implies

$$t = \log 2.$$

(d) Determine the maximum displacement of the object from its equilibrium position.

Solution: We look for critical points

$$0 = y'(t) = -2e^{-2t} + 6e^{-3t},$$

for $t > 0$. We find

$$3 = e^t$$

which implies

$$t = \log 3.$$

This is the only interior critical point. We must compare $y(\log 3)$ to $y(0)$ to determine the maximum displacement.

We compute that

$$y(\log 3) = e^{-2\log 3} - 2e^{-3\log 3} = \frac{1}{9} - \frac{2}{27} = \frac{1}{27}.$$

On the other hand,

$$y(0) = -1.$$

Since the displacement if the absolute value $|y(t)|$, we see that the maximum displacement of $y(t)$ occurs at $t = 0$ and is 1.