MATH 205 PRACTICE EXAM ON LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' + y = 0.$$

Which of the following functions is a solution of the differential equation?

A. $y = \sin(x) + 3\cos(x)$.

B. $y = \sin(3x) + \cos(x)$.

C. $y = e^x + 3e^{-x}$.

D. $y = e^{3x} + e^{-x}$.

E. None of these.

Answer: **A**.

Reason: The auxiliary polynomial is $P(r) = r^2 + 1 = (r + i)(r - i)$. Recalling that $e^{ix} = \cos x + i \sin x$, we deduce that $\{\cos x, \sin x\}$ forms a basis for the space of solutions. (You can also just plug in $y = \sin x + 3\cos x$ to see it satisfies the equation).

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' - y = 0.$$

Which of the following functions is a solution of the differential equation?

A. $y = \sin(x) + 3\cos(x)$.

B. $y = \sin(3x) + \cos(x)$.

C. $y = e^x + 3e^{-x}$.

D. $y = e^{3x} + e^{-x}$.

E. None of these.

Answer: **C**.

Reason: The auxiliary polynomial is $P(r) = r^2 - 1 = (r - 1)(r + 1)$. We deduce that $\{e^x, e^{-x}\}$ forms a basis for the space of solutions. (You can also just plug in $y = e^x + 3e^{-x}$ to see it satisfies the equation).

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' + 100y = 36\cos(8x) + 36\sin(8x).$$

Which of the following functions is a solution of the differential equation?

A. $y = \cos(8x) + \sin(8x)$.

B. $y = 36\cos(8x) + 36\sin(8x)$.

C. $y = \cos(10x) + \sin(10x)$.

D. $y = 36\cos(10x) + 36\sin(10x)$.

E. None of these.

Answer: **A**.

Reason: Compute that if y is given as in A, then $y'' = -64\cos x - 64\sin x$. Thus the answer in A satisfies the ODE. (The RHS of the ODE suggests we should look for a trial solution of the form in A as well.)

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' - 15y' + 56y = 6e^{5x}.$$

Which of the following functions is a solution of the differential equation?

A. $y = e^{4x}$.

B. $y = e^{5x}$.

C. $y = e^{6x}$.

D. $y = e^{7x}$.

E. None of these.

Answer: **B**.

Reason: Compute that if y is given as in B, then $y'' = 25e^{5x}$ and $y' = 5e^{5x}$. Thus the answer in B satisfies the ODE. (The RHS of the ODE suggests we should look for a trial solution of the form in B as well.)

Problem 5. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' - 15y' + 56y = (2 + 2x)e^{8x}.$$

Which of the following functions is a solution of the differential equation?

A. $y = e^{8x}$.

B. $y = xe^{8x}$.

C. $y = x^2e^{8x}$.

D. $y = x^3e^{8x}$.

E. None of these.

Answer: **C**.

Reason: The auxiliary polynomial is $P(r) = r^2 - 15r + 56 = (r - 7)(r - 8)$. Since it has $r = 8$ as a root, we know that $P(D)e^{8x} = 0$. We thus look for a modified trial solution to be of the form $xe^{8x}(c_1 + c_2x)$. This means we should try B or C. In fact, the answer has to be C since we need a nonzero coefficient of xe^{8x} on the RHS and $P(D)$ always annihilates the highest degree x in the trial solution.

Alternatively, we could simply try B first. In that case, we find $y'' = 64xe^{8x} + 8e^{8x}$ and $y' = 8xe^{8x} + e^{8x}$, in which case $y'' - 15y' + 56y = (64 - 8 \cdot 15)xe^{8x} + (8 - 15)e^{8x} + 56xe^{8x} = -7e^{8x}$. So B is not a solution, and we deduce it must be C.

Alternatively, we could observe that

$$\begin{aligned} P(D)(x^m e^{8x}) &= (D - 7)(D - 8)(x^m e^{8x}) \\ &= (D - 7)(mx^{m-1}e^{8x}) \\ &= \Big(m(m - 1)x^{m-2} + mx^{m-1}\Big)e^{8x}. \end{aligned}$$

Pattern matching with the RHS, we deduce that $m = 2$.

Problem 6. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Consider the differential equation

$$y'' - 15y' + 56y = (10 - 10x)e^{7x} + (10 + 10x)e^{8x}.$$

Which of the following functions is a solution of the differential equation?

A. $y = 5x^2(e^{7x} + e^{8x})$.

B. $y = 5x^3(e^{7x} + e^{8x})$.

C. $y = (10 - 10x)e^{7x} + (10 + 10x)e^{8x}$.

D. $y = (10x - 10x^2)e^{7x} + (10x + 10x^2)e^{8x}$.

E. None of these.

Answer: **A**.

Reason: As above, the auxiliary polynomial is $P(r) = r^2 - 15r + 56 = (r - 7)(r - 8)$. Since it has both $r = 7$ and $r = 8$ as roots, we know that $P(D)e^{7x} = P(D)e^{8x} = 0$. We thus look for a modified trial solution to be of the form $xe^{8x}(c_1 + c_2x) + xe^{7x}(c_3 + c_4x)$.. This means we should try A or D. In fact, in the previous MC question, we concluded that $P(D)x^2e^{8x} = (2 + 2x)e^{8x}$. This implies $P(D)(5x^2e^{8x}) = (10 + 10x)e^{8x}$. This means the coefficient for x^2e^{8x} in y is correct in A and not in D, so we conclude A must be the correct answer (by linearity of $P(D)$).

Alternatively, we can check that

$$\begin{aligned} P(D)(5x^2e^{7x}) &= (D - 8)(D - 7)(5x^2e^{7x}) \\ &= (D - 8)(10xe^{7x}) \\ &= (10 - 10x)e^{7x}, \end{aligned}$$

so the coefficient of x^2e^{7x} is also correct in A.

Problem 7. (20 pts.) Consider the differential equation

(1) $$y'' - 20y' + 100y = 6e^{10x}.$$

(a) Find all solutions of the homogeneous differential equation corresponding to (1).

Solution: The auxiliary polynomial if $P(r) = r^2 - 20r + 100 = (r - 10)^2$. The homogeneous equation $P(D)y = 0$ therefore has a general solution given by

$$\boxed{y_c(x) = c_1 e^{10x} + c_2 x e^{10x}.}$$

(b) Find one particular solution of (1).

Solution: Because the right hand side has annihilator polynomial $A(r) = r - 10$ and $r = 10$ is a root of $P(r)$ with multiplicity 2, we consider the modified trial solution

$$y_p(x) = A_0 x^2 e^{10x}.$$

We compute

$$\begin{aligned} P(D)\big(A_0 x^2 e^{10x}\big) &= A_0(D-10)^2(x^2 e^{10x}) \\ &= A_0(D-10)(2xe^{10x}) \\ &= 2A_0 e^{10x}. \end{aligned}$$

Therefore we take $2A_0 = 6$ or $A_0 = 3$. This implies

$$\boxed{y_p(x) = 3x^2 e^{10x}}$$

is a particular solution of (1).

(continued →)

(c) Find all solutions of (1).

Solution: All solutions are of the form $y(x) = y_c(x) + y_p(x)$, so the general solution of the inhomogeneous problem is

$$\boxed{y(x) = c_1 e^{10x} + c_2 x e^{10x} + 3x^2 e^{10x}.}$$

(d) Which solution of (1) satisfies $y(0) = 2$ and $y'(0) = -5$?

Solution: We use the initial conditions to fix the constants c_1, c_2 in the general solution above. Letting $y(x) = c_1 e^{10x} + c_2 x e^{10x} + 3x^2 e^{10x}$, we observe that

$$y'(x) = (10c_1 + c_2)e^{10x} + (10c_2 + 6)xe^{10x} + 30x^2 e^{10x}.$$

Therefore the equations for $y(0) = 2$ and $y'(0) = -5$ correspond to

$$\begin{aligned} c_1 &= 2, \\ 10c_1 + c_2 &= -5. \end{aligned}$$

This gives $c_1 = 2$ and $c_2 = -25$. We conclude the solution of the IVP is

$$\boxed{y(x) = 2e^{10x} - 25xe^{10x} + 3x^2 e^{10x}}$$

Problem 8. (20 pts.) Consider the differential equation

$$y'' + 16y = 27\cos(5x). \tag{2}$$

(a) Find all solutions of the homogeneous differential equation corresponding to (2).

Solution: The auxiliary polynomial if $P(r) = r^2 + 16 = (r - 4i)(r + 4i)$. The homogeneous equation $P(D)y = 0$ therefore has a general solution given by

$$\boxed{y_c(x) = c_1 \cos 4x + c_2 \sin 4x.}$$

(b) Find one particular solution of (2).

Solution: The right hand side has annihilator polynomial $A(r) = r^2 + 25$ which doesn't share any roots with $P(r)$. So we consider the usual trial solution

$$y_p(x) = A_0 \cos 5x + B_0 \sin 5x.$$

We compute

$$\begin{aligned} P(D)\big(A_0 \cos 5x + B_0 \sin 5x.\big) &= A_0(D^2 + 16)\cos 5x + B_0(D^2 + 16)\sin 5x. \\ &= -9A_0 \cos 5x - 9B_0 \sin 5x. \end{aligned}$$

Therefore we take $-9A_0 = 27$ or $-9B_0 = 0$. This implies $A_0 = -3$ and $B_0 = 0$, so

$$\boxed{y_p(x) = -3\cos 5x}$$

is a particular solution of (1).

(continued →)

(c) Find all solutions of (2).

Solution: All solutions are of the form $y(x) = y_c(x) + y_p(x)$, so the general solution of the inhomogeneous problem is

$$\boxed{y(x) = c_1 \cos 4x + c_2 \sin 4x - 3\cos 5x.}$$

(d) Which solution of (2) satisfies $y(0) = -1$ and $y'(0) = -12$?

Solution: We use the initial conditions to fix the constants c_1, c_2 in the general solution above. Letting $y(x) = c_1 \cos 4x + c_2 \sin 4x - 3\cos 5x$, we observe that

$$y'(x) = -4c_1 \sin 4x + 4c_2 \cos 4x + 15 \sin 5x.$$

Therefore the equations for $y(0) = -1$ and $y'(0) = -12$ correspond to

$$\begin{aligned} c_1 - 3 &= -1, \\ 4c_2 &= -12. \end{aligned}$$

This gives $c_1 = 2$ and $c_2 = -3$. We conclude the solution of the IVP is

$$\boxed{y(x) = 2\cos 4x - 3\sin 4x - 3\cos 5x}$$

Problem 9. (20 pts.) Give the general solution to the following differential equations:

(a) $y'''' + 2y'' + y = 0$.

Solution: The auxiliary polynomial is

$$P(r) = r^4 + 2r^2 + 1 = (r^2+1)^2 = (r-i)^2(r+i)^2.$$

It follows that the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

(b) $(D-3)(D^2-2D+5)^2 y = 0$.

Solution: The auxiliary polynomial is

$$P(r) = (r-3)(r^2-2r+5)^2.$$

The roots of the quadratic are

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i,$$

so

$$P(r) = (r-3)\left(r-(1+2i)\right)^2\left(r-(1-2i)\right)^2$$

It follows that the general solution is

$$y(x) = C_1 e^{3x} + C_2 e^x \cos(2x) + C_3 e^x \sin(2x) + C_4 e^x x \cos(2x) + C_5 e^x x \sin(2x).$$

Problem 10. (20 pts.) Consider the following nonhomogeneous differential equation:

$$y''' + 11y'' + 7y' - 147y = -120x^2e^{-7x} + 81e^{2x},$$

which can be equivalently written as $P(D)y = F$:

$$(D+7)^2(D-3)y = -120x^2e^{-7x} + 81e^{2x}.$$

The solution of the homogeneous equations is given by

$$y_c(x) = C_1e^{3x} + C_2e^{-7x} + C_3xe^{-7x}.$$

Find one particular solution of the nonhomogeneous equation.

Solution:

For the nonhomogeneous term $-120x^2e^{-7x}$ (because 7 is a root of multiplicity 2 of $P(r)$), we need to use a modified trial solution of the form

$$y_{p,1}(x) = A_0x^2e^{-7x} + A_1x^3e^{-7x} + A_2x^4e^{-7x}$$

We compute (using the useful identity $(D-\alpha)(x^me^{\alpha x}) = mx^{m-1}e^{\alpha x}$) that

$$\begin{aligned}
P(D)y_{p,1}(x) &= (D-3)(D+7)^2\left(A_0x^2e^{-7x} + A_1x^3e^{-7x} + A_2x^4e^{-7x}\right)\\
&= (D-3)(D+7)\left(2A_0xe^{-7x} + 3A_1x^2e^{-7x} + 4A_2x^3e^{-7x}\right)\\
&= (D-3)\left(2A_0e^{-7x} + 6A_1xe^{-7x} + 12A_2x^2e^{-7x}\right)\\
&= \left((6A_1 - 14A_0)e^{-7x} + (24A_2 - 42A_1)xe^{-7x} - 84A_2x^2e^{-7x}\right)\\
&\quad + \left(-6A_0e^{-7x} - 18A_1xe^{-7x} - 36A_2x^2e^{-7x}\right)\\
&= \left((6A_1 - 20A_0)e^{-7x} + (24A_2 - 60A_1)xe^{-7x} - 120A_2x^2e^{-7x}\right).
\end{aligned}$$

We need this to be equal to $-120x^2e^{-7x}$, so we conclude $A_2 = 1$, hence $24 - 60A_1 = 0 \implies A_1 = \frac{2}{5}$, and $\frac{12}{5} - 20A_0 = 0 \implies A_0 = \frac{3}{25}$. Therefore the particular solution which finds $-120x^2e^{-7x}$ is given by

$$y_{p,1}(x) = \frac{3}{25}x^2e^{-7x} + \frac{2}{5}x^3e^{-7x} + x^4e^{-7x}.$$

Next, for the nonhomogeneous term e^{2x}, we need only look for a trial solution of the form

$$y_{p,2}(x) = B_0e^{2x}.$$

We compute that

$$\begin{aligned}
P(D)y_{p,1}(x) &= (D-3)(D+7)^2(B_0e^{2x})\\
&= (D-3)(D+7)(9B_0e^{2x})\\
&= (D-3)(81B_0e^{2x})\\
&= -81B_0e^{2x}.
\end{aligned}$$

We conclude that $B_0 = -1$ and $y_{p,2}(x) = -e^{2x}$.

Finally, we put $y_{p,1}(x)$ and $y_{p,2}(x)$ together to get a general solution of $P(D)y = -120x^2e^{-7x} + 81e^{2x}$,

$$y_p(x) = \frac{3}{25}x^2e^{-7x} + \frac{2}{5}x^3e^{-7x} + x^4e^{-7x} - e^{2x}.$$