MATH 205 PRACTICE EXAM ON MATRIX ALGEBRA

Name:

Section:

Instructions:

(1) Books, notes, and calculators are not permitted.
(2) Simplify as instructed. On some problems, unsimplified correct answers will not receive full credit.
(3) Show your work as instructed. On some problems, correct answers without justification will not receive full credit.

(1)

(2)

(3)

(4)

(5)

(6)

Total

Problem 1. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A, B, C be $n \times n$ matrices. Which of the following equations is not always true?

A. $A + B = B + A$.

B. $A(B + C) = AB + AC$.

C. $AB = BA$.

D. $(AB)C = A(BC)$.

E. None of these: the four equations hold for all $n \times n$ matrices.

Answer: **C**.
Reason: Matrix multiplication is not necessarily commutative.

Problem 2. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let A, B, C be $n \times n$ matrices. Which of the following equations is not always true?

A. $A^\top + B^\top = (A + B)^\top$.

B. $A(B^\top + C^\top) = AB^\top + AC^\top$.

C. $(AB)^\top = B^\top A^\top$.

D. $(AB)^\top C = A(BC)^\top$.

E. None of these: the four equations hold for all $n \times n$ matrices.

Answer: **D**.
Reason: $(AB)^\top C = B^\top A^\top C \neq AC^\top B^\top = A(BC)^\top$.

Problem 3. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let m, n and p be three distinct positive integers. Let A be an $m \times n$ matrix, let B be an $n \times p$ matrix, and let C be a $p \times m$ matrix. Which of the following products is not defined?

A. AB.

B. AC.

C. BC.

D. ABC.

E. None of these: the four matrices are all defined.

Answer: **B.**
Reason: Since $n = p$, an $m \times n$ matrix cannot left multiply a $p \times m$ matrix.

Problem 4. (5 pts.) Circle the correct answer. There will be no partial credit. You do not need to show your work.

Let m, n and p be three distinct positive integers. Let A be an $m \times n$ matrix, let B be an $n \times p$ matrix, and let C be a $p \times m$ matrix. Which of the following expressions is not defined?

A. $AA^\top + ABC$.

B. $ABC + C^\top C$.

C. $A^\top A + BB^\top$.

D. $ABC + (CBA)^\top$.

E. None of these: the four matrices are all defined.

Answer: **D.**
Reason: As before CB is not defined since $m = n$.

Problem 5. (15 pts.) Find nonzero 2×2 matrices A, B, C with the property that $AB = AC$ but $B \neq C$.

Solution: Observe that $AB = AC$ if and only if $A(B - C) = 0$. Consider two nonzero matrices that mulitply to zero:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now define nonzero matrices

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, by the above $AC = 0$ and additionally $B - C = C$. We conclude $A(B - C) = AC = 0$, hence $AB = AC$. However, $B \neq C$.

Problem 6. (15 pts.) Let $A = (a_{i,j})$ be an $m \times n$ matrix and define the $m \times m$ matrix $B = (b_{i,j})$ by $B = AA^{\top}$.

In what follows, we drop the the comma from matrix entries, i.e. a_{ij} means $a_{i,j}$.

(a) Find a formula for $b_{i,j}$ in terms of the entries of A.

Solution: Recall that if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices such that $B = AC$, then the definition of matrix multiplication gives the identity

$$b_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}.$$

Presently, we have $B = AA^T$, so $C = A^T$ and hence the entries of C are given by $c_{ij} = a_{ji}$. Plugging this in the above, we find

$$\boxed{b_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}.}$$

(b) Explain why $b_{i,j} = b_{j,i}$ for all i, j.

Solution: One way to see this is to know that the expression for b_{ij} above is evidently symmetric in i and j:

$$b_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} = \sum_{k=1}^{n} a_{jk} a_{ik} = b_{ji}.$$

Alternatively,

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

which shows B is symmetric (hence $b_{ij} = b_{ji}$.

Problem 7. (15 pts.) Let $A = (a_{i,j})$ be an $n \times n$ matrix and define

$$B = A + A^{\top}, \qquad C = A - A^{\top}.$$

(a) Explain why $b_{i,j} = b_{j,i}$.

Solution: By properties of the transpose (linearity and involutivity), one has

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B.$$

Hence $b_{ij} = b_{ji}$.

(b) Find a formula for $b_{i,i}$ in terms of the entries of A.

Solution: Since $A = (a_{ij})$, we have $A^T = (a_{ji})$ and so from $B = A + A^T$, in general we have $b_{ij} = a_{ij} + a_{ji}$. Taking $j = i$, we conclude

$$\boxed{b_{ii} = 2a_{ii}.}$$

(a) Explain why $c_{i,j} = -c_{j,i}$.

Solution: By properties of the transpose (linearity and involutivity), one has

$$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C.$$

Hence $c_{ij} = -c_{ji}$.

(b) Find a formula for $c_{i,i}$ in terms of the entries of A.

Solution: From $C = A - A^T$, in general we have $c_{ij} = a_{ij} - a_{ji}$. Taking $j = i$, we conclude

$$\boxed{c_{ii} = 0.}$$

Problem 8. (15 pts.) Define the matrix

$$A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(a) Compute A^2.

Solution: We compute

$$\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4+1/4 & -1/4-1/4 \\ -1/4-1/4 & 1/4+1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

In other words, $\boxed{A^2 = A.}$

(b) Find a formula for A^n, where n is any positive integer.

Solution: If $n = 1$, evidently $A^n = A$. Suppose $n \geq 2$. Then, using part (a) above, we have

$$A^n = A^{n-2}A^2 = A^{n-2}A = A^{n-1}.$$

Therefore, inductively, we have

$$A^n = A^{n-1} = A^{n-2} = \cdots = A^2 = A.$$

Hence $\boxed{A^n = A.}$